

Unit 5

Coordinate geometry and vectors

Introduction

In Unit 2 you saw how lines and parabolas can be represented using equations. The use of equations and the coordinates of points to investigate geometric problems is an area of mathematics known as *coordinate geometry*. In the first part of this unit you will study more of this topic.

A fundamental concept in geometry is that of distance. In Section 1, you will learn how to calculate the distance between two points from their coordinates. You will also discover how to find all the points that are the same distance from one specified point as they are from another.

All the points that are the same distance from a *single* point form a circle. Circles are prevalent throughout our environment, and have been studied since ancient times. A circle is sometimes thought of as the most ‘perfect’ of shapes, since it can be turned through any angle around its centre, or reflected in any line that passes through the centre, and still occupy exactly the same position as the original shape.

In Section 2, you will learn how to find the equations of circles. Section 3 then extends the methods used in Unit 2 for finding the point of intersection of two lines, to obtain methods for finding the points of intersection of lines and other curves, including circles. You will also see how to use a computer to plot circles and some other curves.

All the geometry that you have studied so far in this module has been two-dimensional; that is, it lies within a plane. The world we live in, however, is three-dimensional: objects have length, breadth and depth. Section 4 extends the idea of coordinates to points in a three-dimensional space. You will see how to calculate the distance between two points in three dimensions and how a three-dimensional shape, the sphere, can be represented algebraically. Such representations form the basis of many computer graphics systems.

One way to specify the position of a point, which is often useful in practical situations, is to give both the distance and the direction of the point from a known reference point. For example, in Figure 1 the point B is 2 km north-west of the point A , where north is in the direction indicated.

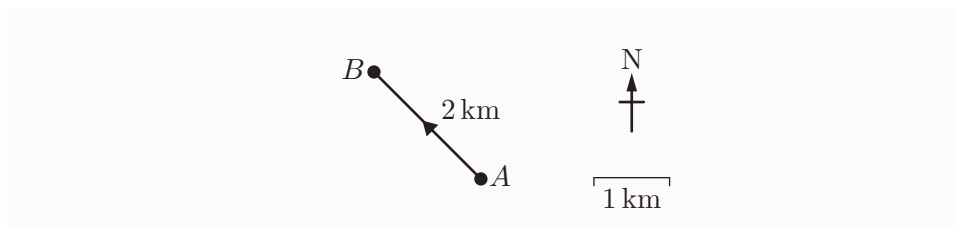


Figure 1 Specifying the position of B relative to A

The distance and direction of a point from a reference point is known as its *displacement*. Quantities such as displacement that have a size *and* a direction are called *vectors*. Vectors have many applications in physics and engineering; for example, specifying the velocity of an object or the force



Circles everywhere!

acting on an object requires the use of vectors. You will study vectors in the second part of this unit, starting in Section 5.

1 Distance

This first section is about finding distances in the plane.

1.1 The distance between two points in the plane

In this subsection you will meet a formula for the distance between any two points in the plane, in terms of their coordinates.

Consider any two points in the plane, say $A(x_1, y_1)$ and $B(x_2, y_2)$. To start with, let's look at the case where B lies above and to the right of A , as illustrated in Figure 2. Let C be the point that lies on the same horizontal line as A and the same vertical line as B , as shown in the figure.

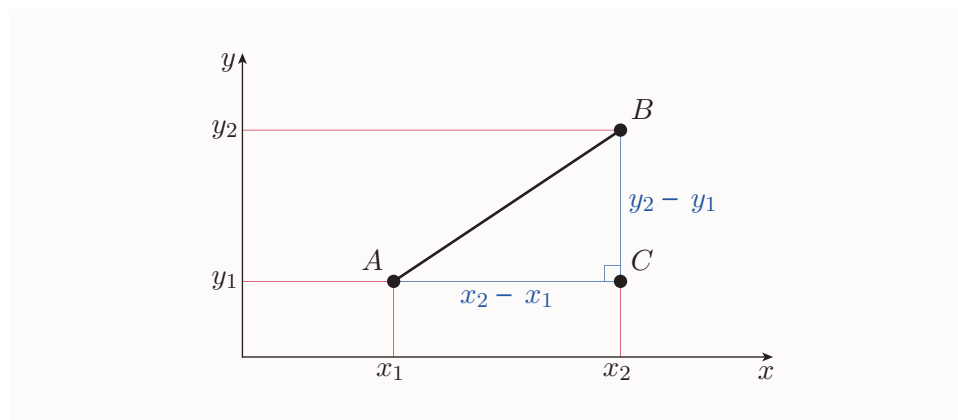


Figure 2 Two points A and B in the plane

The distance between A and B is the length of the hypotenuse of the right-angled triangle with vertices A , B and C . The two shorter sides of this triangle have lengths $x_2 - x_1$ and $y_2 - y_1$, so, by Pythagoras' theorem, the distance between A and B is given by the formula

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \quad (1)$$

In fact, this formula for the distance between A and B holds *no matter where* in the plane A and B lie. For example, suppose that B lies above and to the left of A , as illustrated in Figure 3.

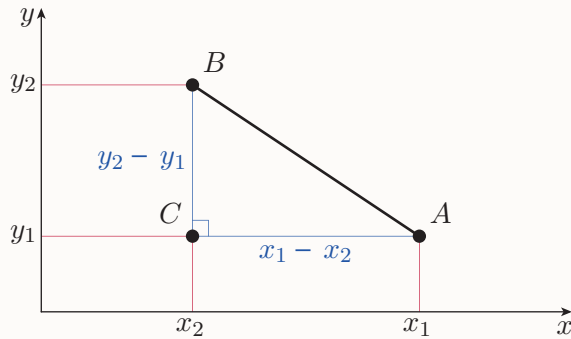


Figure 3 Another pair of points A and B in the plane

In this case, if you consider the point C that lies on the same horizontal line as A and the same vertical line as B , then you obtain a right-angled triangle whose two shorter sides have lengths $x_1 - x_2$ and $y_2 - y_1$. So the distance between A and B is given by

$$\sqrt{(x_1 - x_2)^2 + (y_2 - y_1)^2}.$$

This formula is slightly different from formula (1) (the only difference is that x_1 and x_2 are interchanged), but the two formulas are equivalent. To see this, notice that $x_1 - x_2 = -(x_2 - x_1)$, as you can check by removing the brackets, so $(x_1 - x_2)^2 = (x_2 - x_1)^2$.

You can see that if A and B lie anywhere in the plane, except on the same horizontal or vertical line, then you can form a right-angled triangle whose hypotenuse gives the distance between A and B , and whose two shorter sides have lengths $\pm(x_2 - x_1)$ and $\pm(y_2 - y_1)$. So in all these cases the distance between A and B is given by formula (1).

The formula also holds if A and B lie on the same horizontal or vertical line. For example, if B lies directly to the right of A , as illustrated in Figure 4, then $y_2 - y_1 = 0$, so formula (1) simplifies to $\sqrt{(x_2 - x_1)^2}$. This expression simplifies further to $x_2 - x_1$, since $x_2 - x_1$ is positive, and this is indeed the distance between A and B , as shown in Figure 4.

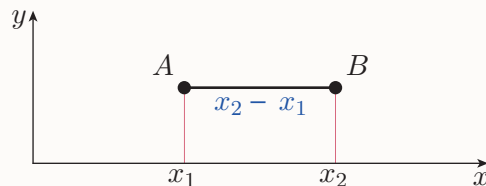


Figure 4 A third pair of points A and B in the plane

In summary, the following result holds in general.

Distance formula

The distance between the points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 1 *Finding the distance between two points*

Find the distance between the points $(2, 4)$ and $(5, 3)$.

Solution

Take $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (5, 3)$ in the formula above.

The distance is

$$\begin{aligned}\sqrt{(5 - 2)^2 + (3 - 4)^2} &= \sqrt{3^2 + (-1)^2} \\ &= \sqrt{9 + 1} \\ &= \sqrt{10}.\end{aligned}$$

Leave the answer as a surd, since a decimal answer was not requested.

Activity 1 *Finding the distance between two points*

Find the distance between the two points in each of the following pairs.

- (a) $(3, 2)$ and $(7, 5)$ (b) $(-1, 4)$ and $(3, -2)$

1.2 Midpoints and perpendicular bisectors

In coordinate geometry, it is often useful to consider the collection of all the points that satisfy a particular property. You have already met some examples of this. For example, you have seen that the collection of all points whose y -coordinate is twice their x -coordinate is described by the equation $y = 2x$. The collection of all points that satisfy a particular property is called a **locus** of points. 'Locus' is the Latin word for 'place'.

Sometimes it is useful to consider the locus of points that satisfy a property involving distance. In the next activity you are asked to consider the locus of points that are **equidistant** from (the same distance from) each of two particular points.

Activity 2 *Finding points equidistant from two other points*

Mark two points on a piece of paper. Draw a third point such that its distance from each of the original points is the same. Then mark some more points with this property. Can you describe the geometrical object formed by all the points with this property?

The answer to Activity 2 is that all the points that are equidistant from the two original points form a straight line, which passes through the point halfway along the line segment that joins the two original points, and is perpendicular to this line segment. In the rest of this subsection you'll see one way to find the equation of this line, from the coordinates of the two original points.

The first step is to find the coordinates of the point halfway along the line segment that joins the two original points. This point is called the **midpoint** of the line segment.

Consider the midpoint M of the line segment joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$, as shown in Figure 5. The figure shows B below and to the right of A , but the following argument applies in all cases.

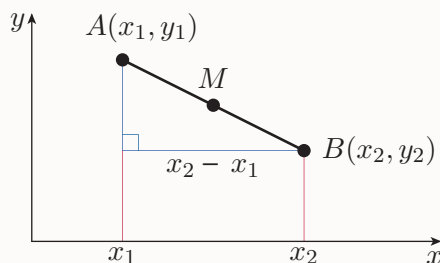


Figure 5 The midpoint M of the line segment AB

You can see from Figure 5 that the x -coordinate of M is equal to the x -coordinate of A plus half of the run from A to B . As you saw in Unit 2, the run from A to B is the value $x_2 - x_1$. It can be positive, negative or zero, depending on whether B lies to the right or left of A , or neither.

So the x -coordinate of M is

$$x_1 + \frac{x_2 - x_1}{2} = \frac{2x_1 + x_2 - x_1}{2} = \frac{x_1 + x_2}{2}.$$

A similar argument applies to the y -coordinate, so we have the following result.

Midpoint formula

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

That is, the x -coordinate of the midpoint is the mean of the x -coordinates of the two points, and the y -coordinate of the midpoint is the mean of the y -coordinates of the two points.

Activity 3 *Calculating the coordinates of a midpoint*

Find the midpoint of the line segment joining the points $(3, 4)$ and $(5, -3)$.

Earlier in this subsection you saw that if you choose any two points, then all the points equidistant from these two points form a straight line that passes through the midpoint of the line segment that joins the two points, and is perpendicular to this line segment.

The line that is perpendicular to a line segment and divides it into two equal parts is called its **perpendicular bisector**. (The word ‘bisect’ means to divide into two equal parts.) For example, Figure 6 shows the perpendicular bisector of a line segment AB .

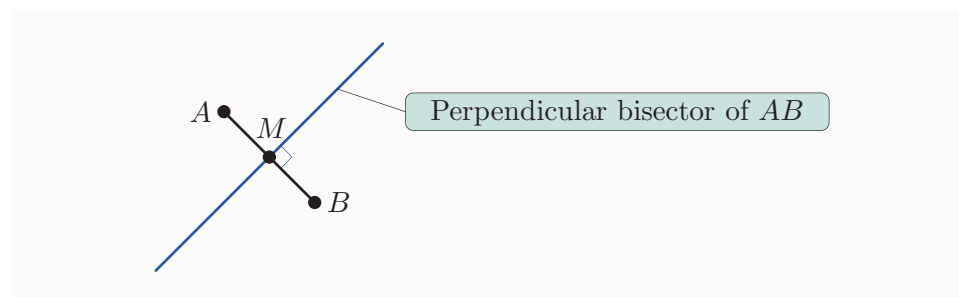


Figure 6 The midpoint M and perpendicular bisector of AB

In Subsection 2.3 of Unit 2, you saw that if two lines are perpendicular and are not parallel to the axes, then the product of their gradients is -1 . So if A and B are points that do not lie on the same horizontal or vertical line, then the gradient of the perpendicular bisector of AB is

$$\frac{-1}{\text{gradient of } AB}.$$

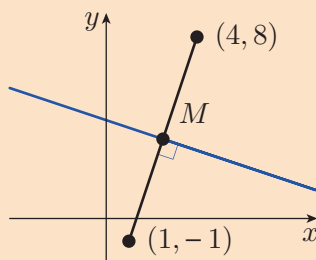
You can use this fact to help you find the equation of a perpendicular bisector, as demonstrated in the next example.

Example 2 Finding a perpendicular bisector

Find the equation of the perpendicular bisector of the line segment joining the points $(4, 8)$ and $(1, -1)$.

Solution

Draw a diagram showing the given points, the midpoint and the perpendicular bisector.



Find the midpoint of the line segment.

The midpoint, M , has coordinates

$$\left(\frac{4+1}{2}, \frac{8+(-1)}{2} \right) = \left(\frac{5}{2}, \frac{7}{2} \right).$$

Find the gradient of the perpendicular bisector, by first finding the gradient of the line segment. Remember that the gradient of the line through (x_1, y_1) and (x_2, y_2) is $(y_2 - y_1)/(x_2 - x_1)$.

The gradient of the line segment is $\frac{-1-8}{1-4} = \frac{-9}{-3} = 3$.

So the gradient of the perpendicular bisector is $-1/3 = -\frac{1}{3}$.

Now find the equation of the perpendicular bisector. Use the fact that the line with gradient m passing through the point (x_1, y_1) has equation $y - y_1 = m(x - x_1)$.

Since M has coordinates $(\frac{5}{2}, \frac{7}{2})$, the perpendicular bisector has equation

$$y - \frac{7}{2} = -\frac{1}{3} \left(x - \frac{5}{2} \right).$$

Simplifying gives

$$\begin{aligned} y - \frac{7}{2} &= -\frac{1}{3}x + \frac{5}{6} \\ y &= -\frac{1}{3}x + \frac{13}{6}. \end{aligned}$$

So the equation of the perpendicular bisector is $y = -\frac{1}{3}x + \frac{13}{6}$.



Here is an example for you to try.

Activity 4 Finding a perpendicular bisector

Find the equation of the perpendicular bisector of the line segment joining $(-3, 1)$ and $(1, -1)$.

There are often alternative ways to solve problems in coordinate geometry. For example, you can find the perpendicular bisector of the line segment joining two points, (a, b) and (c, d) say, as follows. The points (x, y) on this bisector are exactly those that are equidistant from (a, b) and (c, d) ; that is, those satisfying the equation $\sqrt{(x-a)^2 + (y-b)^2} = \sqrt{(x-c)^2 + (y-d)^2}$. So this is the equation of the perpendicular bisector. You can simplify it to obtain the equation of a line in the usual form.

You will use perpendicular bisectors to help you find the equations of circles in Subsection 2.2.

2 Circles

A **circle** can be defined as the locus of points that are at a particular distance from a specified point. The specified point is the **centre** of the circle, and the distance is its **radius**.

This definition of a circle corresponds to the usual way of drawing one, by using a pair of compasses, or by fixing one end of a taut piece of string and marking the curve swept out by the other end.

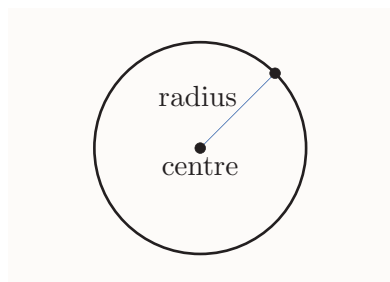
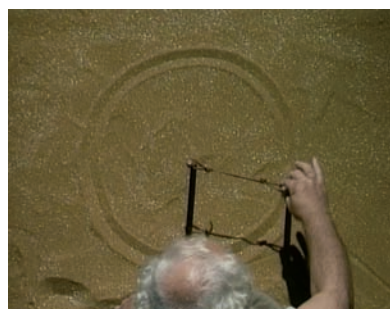


Figure 7 A circle

2.1 The equation of a circle

Just as lines and parabolas in the plane can be represented by equations, so can circles. For example, consider the circle with centre $C(4, 3)$ and radius 2 which is shown in Figure 8.



Marking out a circle in sand

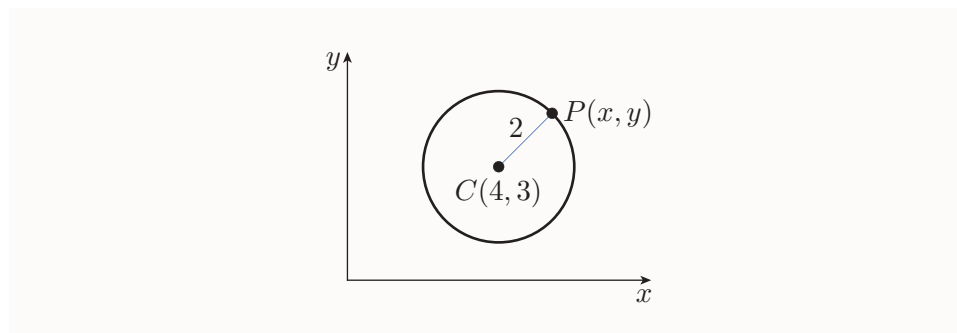


Figure 8 The circle with centre $(4, 3)$ and radius 2

Suppose that $P(x, y)$ is any point on this circle. The distance between the centre and P is equal to the radius, so, by the distance formula,

$$\sqrt{(x - 4)^2 + (y - 3)^2} = 2.$$

Squaring both sides of this equation, to remove the square root, gives the equivalent equation

$$(x - 4)^2 + (y - 3)^2 = 4.$$

Since this equation is satisfied by every point (x, y) on the circle, and only by these points, it is the equation of the circle.

In general, consider the circle with centre (a, b) and radius r , as illustrated in Figure 9.

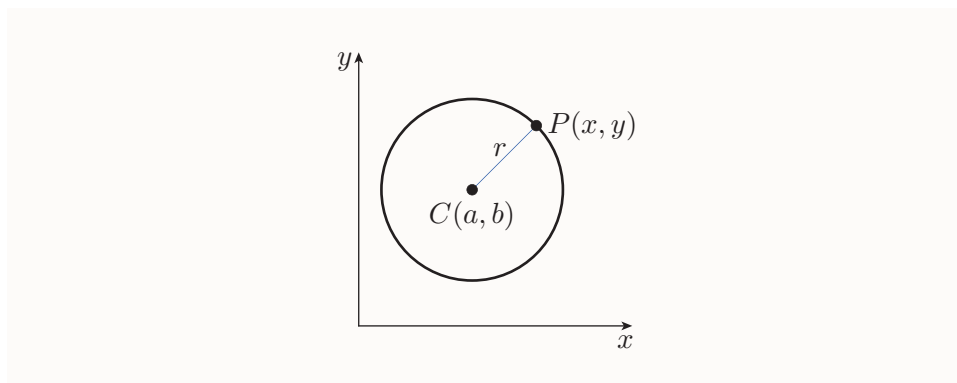


Figure 9 The circle with centre (a, b) and radius r

By the distance formula, every point (x, y) on the circle satisfies the equation

$$\sqrt{(x - a)^2 + (y - b)^2} = r.$$

Squaring this equation to remove the square root gives the **standard form of the equation of a circle** stated below. You will see other forms of this equation later.

Standard form of the equation of a circle

The circle with centre (a, b) and radius r has equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

Example 3 *Finding the equation of a circle*

Find the equation of the circle with centre $(2, -1)$ and radius 3.

Solution

 Use the fact in the box above, with $(a, b) = (2, -1)$ and $r = 3$. 

The equation is

$$(x - 2)^2 + (y - (-1))^2 = 3^2;$$

that is,

$$(x - 2)^2 + (y + 1)^2 = 9.$$

Activity 5 *Writing down equations of circles*

Write down the equations of the circles with the following centres and radii.

(a) Centre $(0, 0)$, radius 3 (b) Centre $(5, 7)$, radius $\sqrt{2}$

(c) Centre $(-3, -1)$, radius $\frac{1}{2}$ (d) Centre $(3, -\sqrt{3})$, radius $\frac{2}{\sqrt{5}}$

If you have the equation of a circle in standard form, then you can write down the centre and radius of the circle immediately.

Example 4 *Finding the centre and radius of a circle*

Find the centre and radius of the circle with equation

$$(x + 3)^2 + (y - 2)^2 = 16.$$

Solution

 Comparing this particular equation with the general equation $(x - a)^2 + (y - b)^2 = r^2$ gives $a = -3$, $b = 2$ and $r^2 = 16$. Remember that r is *positive*, since it is a length. 

The circle has centre $(-3, 2)$ and radius $\sqrt{16} = 4$.

Here are some examples for you to try.

Activity 6 *Finding centres and radii of circles*

Write down the centre and radius of the circle represented by each of the following equations.

(a) $(x - 1)^2 + (y - 2)^2 = 25$ (b) $(x + 1)^2 + (y + 2)^2 = 49$

(c) $(x - \pi)^2 + (y + \pi)^2 = \pi^2$ (d) $4x^2 + 4(y - \sqrt{3})^2 = 7$

You saw in Units 1 and 2 that the values that satisfy an equation do not change when the equation is rearranged into an equivalent form. So the points that satisfy the equation of a circle do not change when the equation is rearranged; that is, the circle represented does not change.

For example, consider the equation of the circle with centre $(4, 3)$ and radius 2, which is

$$(x - 4)^2 + (y - 3)^2 = 4.$$

Expanding the brackets gives

$$x^2 - 8x + 16 + y^2 - 6y + 9 = 4,$$

and simplifying gives

$$x^2 + y^2 - 8x - 6y + 21 = 0.$$

So this is an alternative form of the equation of the circle with centre $(4, 3)$ and radius 2.

If you have the equation of a circle written in an alternative form like this, then you can find the centre and radius of the circle by rearranging the equation into the standard form $(x - a)^2 + (y - b)^2 = r^2$. This allows you to read off the centre and radius in the usual way. To manipulate the equation into standard form, you can use the technique of completing the square, which you revised in Subsection 4.4 of Unit 2. The process is illustrated in the following example.

Example 5 *Finding the centre and radius of a circle whose equation is not in standard form*

Show that the equation

$$x^2 + y^2 + 4x - 12y - 9 = 0$$

represents a circle, and find its centre and radius.

Solution

 Aim to express the equation in standard form. First concentrate on the subexpression formed by the terms in x^2 and x , and complete the square in this subexpression. 



The equation can be rearranged as

$$x^2 + 4x + y^2 - 12y - 9 = 0.$$

Completing the square gives

$$\boxed{x^2 + 4x} + y^2 - 12y - 9 = 0$$

$$\boxed{(x + 2)^2 - 4} + y^2 - 12y - 9 = 0$$

 Then do the same for the subexpression formed by the terms in y^2 and y . 

$$(x + 2)^2 - 4 + \boxed{y^2 - 12y} - 9 = 0$$

$$(x + 2)^2 - 4 + \boxed{(y - 6)^2 - 36} - 9 = 0$$

 Gather the constants on the right, and combine them. 

$$(x + 2)^2 + (y - 6)^2 = 49.$$

This is the equation of a circle in standard form, so the original equation represents a circle.

The centre is $(-2, 6)$ and the radius is $\sqrt{49} = 7$.

Activity 7 Finding centres and radii of circles whose equations are not in standard form

Show that each of the following equations represents a circle, and find the centre and radius in each case.

(a) $x^2 + y^2 - 6x + 8y = 0$ (b) $4x^2 + 4y^2 - 16x + 4y = 3$

You have seen that the equation of any circle can be written as an equation with terms in x^2 , y^2 , x and y and a constant term. However, not every equation of this form is the equation of a circle, because not every equation of this form can be rearranged into the form

$$(x - a)^2 + (y - b)^2 = r^2, \tag{2}$$

for some constants a , b and r .

First, any equation with terms as described above in which x^2 and y^2 have *different* coefficients is not the equation of a circle. This is because when you multiply out an equation of form (2), you always obtain an equation in which the coefficients of x^2 and y^2 are both 1, so if you multiply the equation through by a constant, then x^2 and y^2 always have the *same*

coefficient. For example, the equation $3x^2 + 2y^2 + 4x + 4y - 5 = 0$ is not the equation of a circle.

Even if the coefficients of x^2 and y^2 are the same in an equation with terms as described above, it does not guarantee that the equation represents a circle. For example, consider the equation $x^2 + y^2 - 4x + 2y + 9 = 0$.

Rearranging this equation by completing the squares gives

$(x - 2)^2 + (y + 1)^2 = -4$. No point (x, y) satisfies this equation, because the square of every real number is non-negative. So this equation does not represent any curve. Similarly, the equation $(x - 5)^2 + (y - 3)^2 = 0$ is satisfied only by the single point $(5, 3)$ and hence is not the equation of a circle, since a single point is not usually regarded as a circle.

Some equations with terms in x^2 , y^2 , x and y and a constant term represent curves other than circles. For example, the equation $x^2 + 2y^2 = 4$ represents an *ellipse*, and the equation $x^2 - 2y^2 = 4$ represents a *hyperbola*. These curves are shown in Figure 10. You can learn more about ellipses and hyperbolas in the module *Essential mathematics 2* (MST125).

Activity 8 Identifying the equations of circles

Which of the following equations represent a circle? For each equation that represents a circle, find the centre and radius.

- (a) $(x - 2)^2 - (y + 3)^2 = 4$ (b) $(x + 1)^2 + (y - 2)^2 + 9 = 0$
 (c) $(x + 1)^2 + (y - 3)^2 - 5 = 0$ (d) $2x^2 + 2y^2 - 20x + 16y + 90 = 0$
 (e) $x^2 + y^2 - 10x - 2y + 20 = 0$ (f) $2x^2 + 3y^2 - 5x + 4y - 8 = 0$

Finding points that lie on a given circle is not as straightforward as finding points that lie on a line or a parabola, as you did in Unit 2, since the equation of a circle does not express either variable x or y as a function of the other variable.

However, you can substitute any particular value of x into the equation of a circle, and solve the resulting equation to find the corresponding values of y . The equation that you have to solve is quadratic, so you obtain two, one or no values of y , and each of these values gives a point on the circle. Similarly, you can substitute any particular value of y into the equation of a circle to find two, one or no corresponding values of x , and each of these values gives a point on the circle. Here's an example.

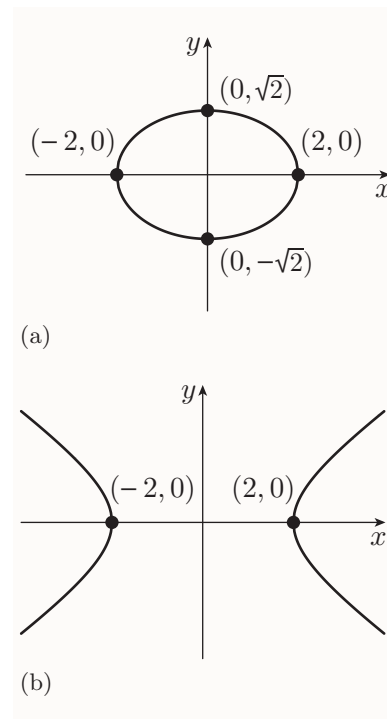


Figure 10 (a) The ellipse $x^2 + 2y^2 = 4$ (b) the hyperbola $x^2 - 2y^2 = 4$

Example 6 *Finding points on a circle*

Find the points (if there are any) on the circle $x^2 + y^2 - 2x - 6y + 8 = 0$ with each of the following x -coordinates.

- (a) 0 (b) 4

Solution

 In each case, substitute the value of x into the equation of the circle, and solve for y . 

- (a) Putting $x = 0$ gives

$$\begin{aligned}y^2 - 6y + 8 &= 0 \\(y - 4)(y - 2) &= 0.\end{aligned}$$

So $y = 4$ or $y = 2$.

The points with x -coordinate 0 on the circle are (0, 2) and (0, 4).

- (b) Putting $x = 4$ gives

$$\begin{aligned}4^2 - 2 \times 4 + y^2 - 6y + 8 &= 0 \\y^2 - 6y + 16 &= 0.\end{aligned}$$

 Remember that the quadratic equation $ax^2 + bx + c = 0$ has no solutions if its discriminant $b^2 - 4ac$ is negative. 

The discriminant of this equation is $6^2 - 4 \times 1 \times 16 = -28$.

Hence the equation has no solutions, so there are no points on the circle with x -coordinate 4.

Note that we often shorten a phrase such as ‘the circle with equation $x^2 + y^2 - 2x - 6y + 8 = 0$ ’ to just ‘the circle $x^2 + y^2 - 2x - 6y + 8 = 0$ ’, as in Example 6.

You have seen that when you substitute a particular value of x into the equation of a circle and solve for y , you obtain two, one or no solutions. If there are two solutions, then one of them gives a point on the top half of the circle, and the other gives a point on the bottom half. If there is only one solution, then it gives the point at the extreme left or right of the circle. If there are no solutions, then there is no point on the circle with that value of x .

These three cases are illustrated in Figure 11(a). Similar cases occur when you substitute a particular value of y into the equation of a circle and solve for x , as illustrated in Figure 11(b).

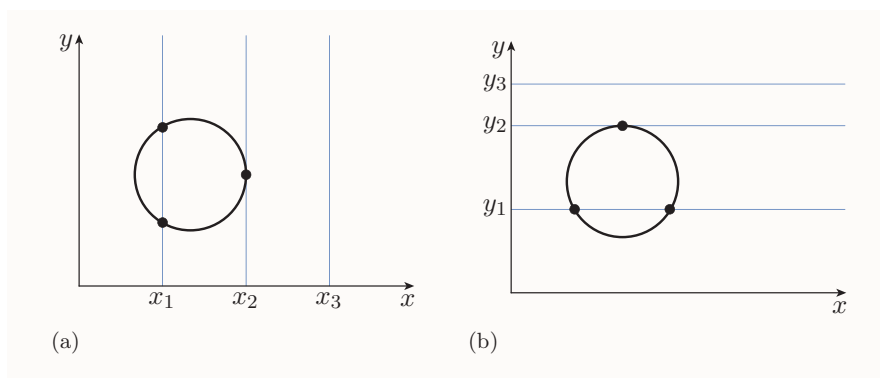


Figure 11 (a) There are two points on the circle corresponding to $x = x_1$, one point corresponding to $x = x_2$ and no points corresponding to $x = x_3$ (b) similar cases for three values of y

Activity 9 Finding points on a circle

- (a) Find all points with y -coordinate 3 on the circle
 $(x - 1)^2 + (y - 5)^2 = 9$.
- (b) Find all points with x -coordinate 4 on the same circle.

The equation of a circle, such as

$$(x - 1)^2 + (y - 5)^2 = 9, \quad (3)$$

is said to be in **implicit form**, because it doesn't express either of the two variables x and y explicitly in terms of the other variable.

By contrast, an equation such as

$$y = 3x^2 + 2,$$

which expresses the variable y explicitly as a function of the other variable x , is said to be in **explicit form**. Some equations in the variables x and y can be written in either implicit or explicit form. For example, the equation $3y + 2x = 5$, which is in implicit form, can be rewritten in explicit form as $y = \frac{1}{3}(5 - 2x)$.

The equation of a circle, however, can't be written in explicit form. Although it's possible to rearrange the equation of a circle to obtain the variable y by itself on the left-hand side, you can't rearrange it to express y as a *function* of x . For example, you can rearrange equation (3) as follows:

$$\begin{aligned} (x - 1)^2 + (y - 5)^2 &= 9 \\ (y - 5)^2 &= 9 - (x - 1)^2 \\ y - 5 &= \pm \sqrt{9 - (x - 1)^2} \\ y &= 5 \pm \sqrt{9 - (x - 1)^2}. \end{aligned}$$

This equation doesn't express y as a function of x , because each value of x gives *two* values of y , due to the $\pm$ sign.

2.2 The circle through three points

In Unit 2, you saw that if you choose any two points, then there is exactly one straight line that passes through both of them. This is not true of circles: there are many circles that pass through two chosen points, as illustrated in Figure 12.

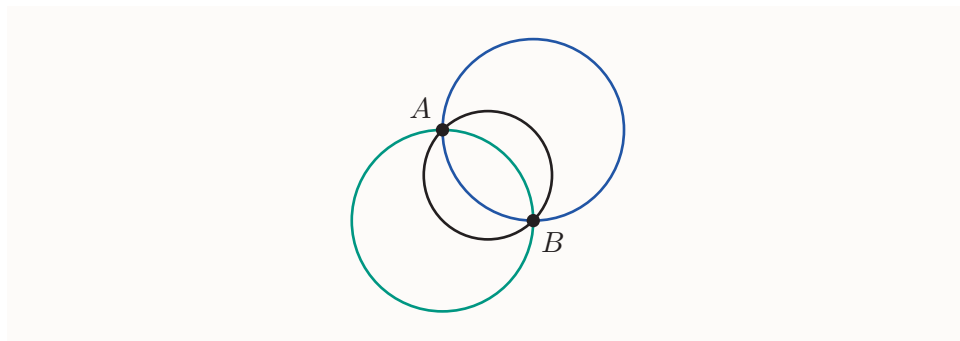


Figure 12 Circles passing through the points A and B

However, if you choose a third point, then as long as the three points do not all lie on the same straight line, there is only one circle that passes through all three points. If the three points *do* all lie on a straight line, then there is no circle that passes through them all. In this subsection you will see how to find the equation of the circle that passes through three chosen points.

Consider three points A , B and C , as illustrated in Figure 13. To find the equation of the circle passing through them, all you need to do is find the coordinates of the centre, labelled D in Figure 13. This is because once you know the coordinates of the centre, it is straightforward to calculate the radius, since it is the distance between the centre and any one of the points A , B and C . So how can you find the coordinates of the centre?

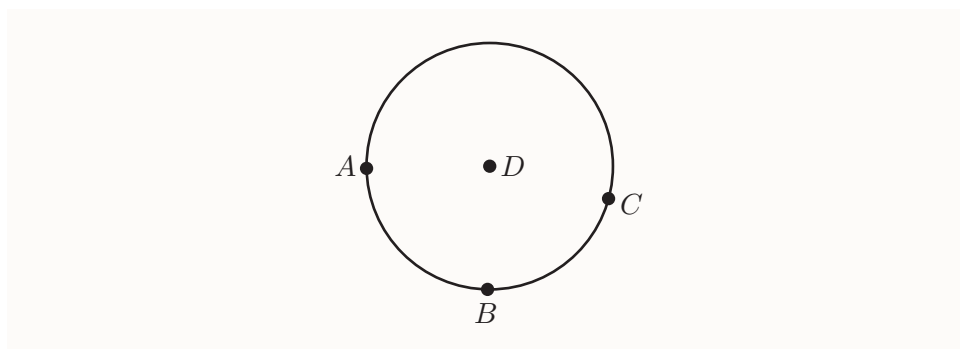


Figure 13 Three points A , B and C determine a circle

The centre of the circle is equidistant from A , B and C . In particular, it is equidistant from A and B , so it lies on the perpendicular bisector of AB . Similarly, it is equidistant from B and C , so it also lies on the

perpendicular bisector of BC . It is the point at which the two perpendicular bisectors intersect. This is illustrated in Figure 14.

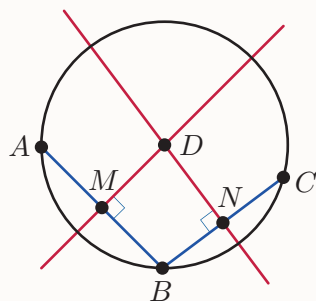


Figure 14 The centre of the circle is the point of intersection of the perpendicular bisectors of AB and BC

So you can use the following strategy to find the equation of the circle passing through three specified points.

Strategy:

To find the equation of the circle passing through three points

1. Find the equation of the perpendicular bisector of the line segment that joins any pair of the three points.
2. Find the equation of the perpendicular bisector of the line segment that joins a different pair of the three points.
3. Find the point of intersection of these two lines. This is the centre of the circle.
4. Find the radius of the circle, which is the distance from the centre to any of the three points.

Hence write down the equation of the circle.

The strategy is illustrated in Example 7 below.

Example 7 Finding the circle through three points

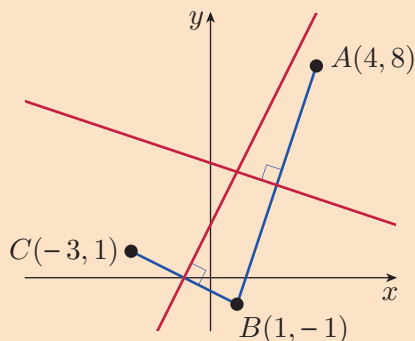
Find the centre and radius of the circle that passes through the points $(4, 8)$, $(1, -1)$ and $(-3, 1)$. Hence write down the equation of the circle.



Solution

Let the points be $A(4, 8)$, $B(1, -1)$ and $C(-3, 1)$.

First sketch a diagram, showing the given points and the perpendicular bisectors.



Find the perpendicular bisectors of the line segments joining two pairs of points. The perpendicular bisector of AB was found in Example 2 on page 109, and you were asked to find the perpendicular bisector of BC in Activity 4 on page 110.

The perpendicular bisector of AB is $y = -\frac{1}{3}x + \frac{13}{3}$.

The perpendicular bisector of BC is $y = 2x + 2$.

The centre of the circle lies on both these lines. To find it, solve the equations simultaneously, as explained in Subsection 3.1 of Unit 2.

The two expressions for y give

$$-\frac{1}{3}x + \frac{13}{3} = 2x + 2.$$

Simplifying gives

$$\frac{7}{3} = \frac{7}{3}x,$$

so $x = 1$. Substituting into $y = 2x + 2$ gives $y = 2 \times 1 + 2 = 4$.

The centre of the circle is $(1, 4)$.

The radius is the distance between the centre and any one of the given points.

The radius is the distance between the centre and $(4, 8)$, which is

$$\sqrt{(4-1)^2 + (8-4)^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$$

Finally, write down the equation of the circle.

The equation of the circle is $(x-1)^2 + (y-4)^2 = 25$.

You can check the answer to Example 7 by confirming that each of the points A , B and C satisfies the equation found. Figure 15 shows the circle whose equation was found in Example 7.

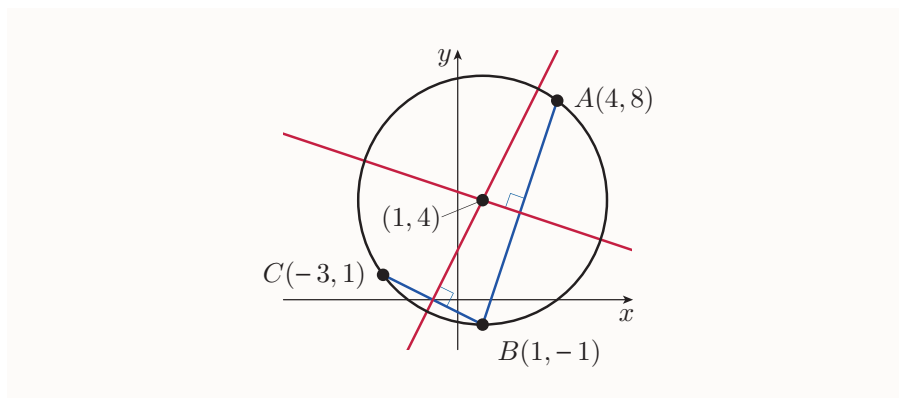


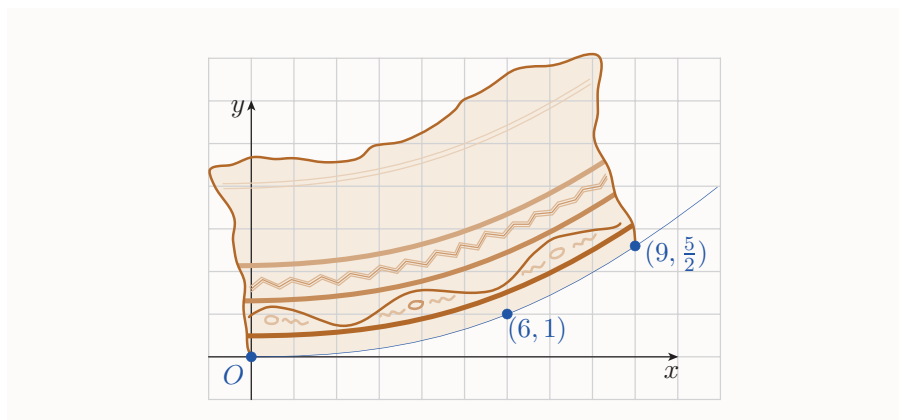
Figure 15 The circle in Example 7

Activity 10 Finding the circle through three points

Find the equation of the circle that passes through the points $(0, 0)$, $(-4, 2)$ and $(8, 6)$.

Activity 11 Finding the diameter of a plate from a fragment

An archaeologist has uncovered a fragment of an ancient plate that she believes was circular. To help her find the diameter of the plate, she places a 1 cm grid over the fragment, as shown below.



The outer edge of the fragment passes through the points $(0, 0)$, $(6, 1)$ and $(9, \frac{5}{2})$. Use this information to find the diameter of the plate.

3 Points of intersection

From Unit 2, you know that a point where two lines intersect can be found by solving the equations of the lines simultaneously. You can use a similar approach to find points where a line and a curve, such as a circle, intersect.

3.1 Points of intersection of a line and a curve

A line and a circle can have two, one or no points of intersection, as illustrated in Figure 16.

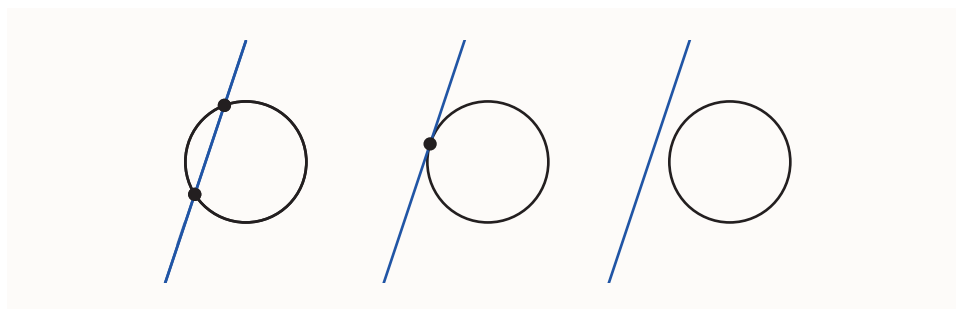


Figure 16 A line can intersect a circle twice, once or not at all

To find the points of intersection, you need to solve the equation of the line and the equation of the circle simultaneously. This leads to a *quadratic* equation, and the three cases that arise in solving such an equation (two solutions, one solution or no solutions) correspond to the three possibilities shown in Figure 16. The method is demonstrated in Example 8 below.



Example 8 Finding the points of intersection of a line and a circle

Find any points at which the line $y = 2x - 4$ intersects the circle $(x + 7)^2 + (y - 2)^2 = 80$.

Solution

The equation of the line gives y in terms of x , so use it to substitute for y in the equation of the circle.

Substituting $y = 2x - 4$ into the equation of the circle gives

$$(x + 7)^2 + (2x - 4 - 2)^2 = 80$$

☁ Simplify. ☁

$$(x + 7)^2 + (2x - 6)^2 = 80$$

$$x^2 + 14x + 49 + 4x^2 - 24x + 36 = 80$$

$$5x^2 - 10x + 5 = 0$$

$$x^2 - 2x + 1 = 0.$$

☁ Solve this quadratic equation. ☁

Factorising gives

$$(x - 1)^2 = 0$$

$$x - 1 = 0,$$

so the only solution is $x = 1$.

☁ Find the corresponding value of y from the equation of the line. ☁

Substituting in $y = 2x - 4$ gives $y = 2 \times 1 - 4 = -2$.

There is only one point of intersection, namely $(1, -2)$.

☁ As a check, you can confirm that the point $(1, -2)$ satisfies the equations of the line and the circle. ☁

Figure 17 shows the point of intersection of the line and the circle in Example 8.

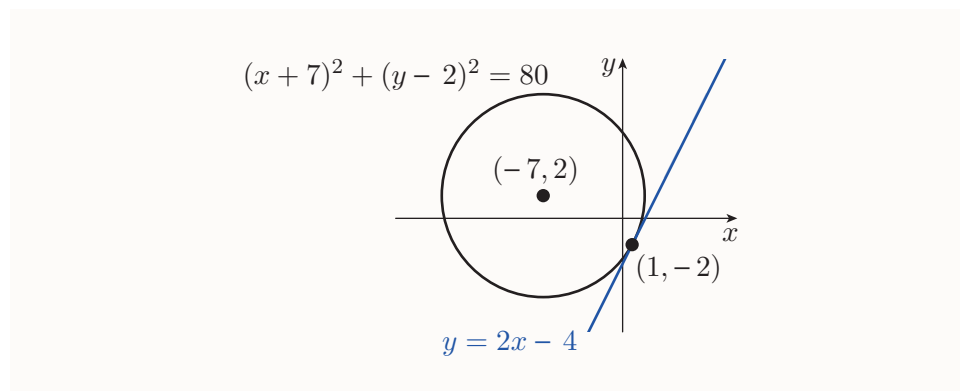


Figure 17 The line and circle in Example 8

Activity 12 *Finding the points of intersection of a line and a circle*

For each of the following lines, either find the point or points at which the line intersects the circle $(x + 7)^2 + (y - 2)^2 = 80$, or show that the line does not intersect the circle.

- (a) $y = 2x$ (b) $y = 2x - 6$

You can use similar methods to find the points of intersection of a line and a parabola. (You studied parabolas in Unit 2.) As in the case of a line and a circle, a line and a parabola may intersect twice, once, or not at all, as illustrated in Figure 18.



Figure 18 A line can intersect a parabola twice, once or not at all

Activity 13 *Finding the points of intersection of a line and a parabola*

Find the points (if there are any) at which the line $y = x + 11$ intersects the parabola $y = -x^2 - 2x + 15$.

Supply and demand

In most economic models, the price of goods is assumed to be determined by the laws of supply and demand.

The *law of demand* states that the lower the selling price of the goods, the more consumers want to buy.

The *law of supply* states that the lower the selling price of the goods, the less producers want to produce.

The law of supply holds because when the price of a product is low, producers can make more profit by using the same materials and labour to produce different, higher-value products.

The selling price at which the supply of a particular item from producers is equal to the demand from consumers is called the **equilibrium price** for the item. Above this price, consumers buy less than sellers want to produce, so the price falls. Below this price consumers want to buy more than sellers produce, so the price rises.

Activity 14 *Working with supply and demand*

A commodity trader wishes to forecast the market price of wheat for the following year. Market research indicates that the relationship between the market demand and selling price for wheat can be modelled by the equation

$$p = 500 - 30q,$$

where q is the amount of wheat expected to be bought (in millions of tonnes), and p is the price of wheat (in £ per tonne).

The trader believes that the relationship between the market supply and selling price of wheat can be modelled by the equation

$$p = \frac{q^2}{2} + \frac{q}{2},$$

where q is the amount of wheat expected to be produced (in millions of tonnes) and p is the price of wheat (in £ per tonne).

Find the equilibrium price of wheat (to the nearest pound) and the quantity expected to be produced (to the nearest million tonnes) in the following year.



The price of wheat is often used to illustrate economic laws

3.2 Points of intersection of two circles

Sometimes it's useful to find the points where two curves, such as two circles, intersect.

Two circles may intersect at two points, one point or not at all, as illustrated in Figure 19.

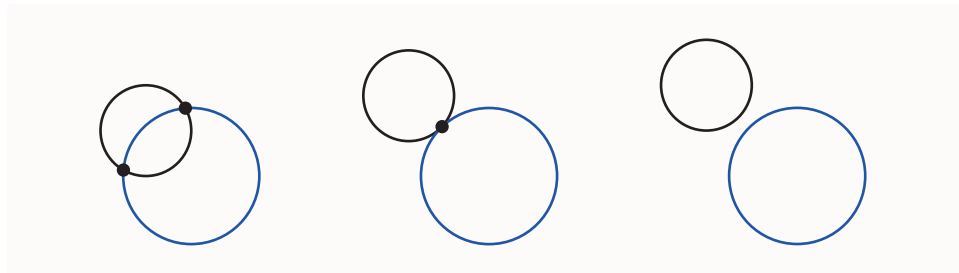


Figure 19 Two circles can intersect twice, once or not at all

To find the points of intersection of two circles, you need to solve their equations simultaneously. It's not straightforward to use one equation to find an expression for x or y to substitute into the other equation, because both equations have terms in x^2 and y^2 . The next example illustrates a convenient method, based on first eliminating the terms in x^2 and y^2 by subtracting one equation from the other (or a multiple of one equation from a multiple of the other). This is always possible for the equations of

two circles, because in the equation of a circle, x^2 and y^2 always have the same coefficient.



Example 9 Finding the points of intersection of two circles

Find the points of intersection (if there are any) of the circles with equations

$$(x - 3)^2 + (y + 4)^2 = 53 \quad \text{and} \quad (x + 2)^2 + (y - 1)^2 = 13.$$

Solution

First multiply out the brackets in each equation and simplify it.

The equations can be written as

$$x^2 - 6x + 9 + y^2 + 8y + 16 = 53$$

$$x^2 + 4x + 4 + y^2 - 2y + 1 = 13;$$

that is,

$$x^2 + y^2 - 6x + 8y - 28 = 0 \tag{4}$$

$$x^2 + y^2 + 4x - 2y - 8 = 0. \tag{5}$$

Eliminate the terms in x^2 and y^2 , by subtracting one equation from the other.

Subtracting equation (5) from equation (4) gives

$$-10x + 10y - 20 = 0;$$

that is,

$$y = x + 2.$$

This tells you that if the point (x, y) satisfies both equation (4) and equation (5), then it also satisfies the equation $y = x + 2$.

Using this equation to substitute for y in equation (4) gives

$$x^2 + (x + 2)^2 - 6x + 8(x + 2) - 28 = 0$$

$$2x^2 + 6x - 8 = 0$$

$$x^2 + 3x - 4 = 0.$$

Solve this quadratic equation.

Factorising gives

$$(x + 4)(x - 1) = 0,$$

so the solutions are $x = -4$ and $x = 1$.

Use the equation $y = x + 2$ to find the corresponding values of y .

The corresponding values of y are

$$y = -4 + 2 = -2 \quad \text{and} \quad y = 1 + 2 = 3.$$

So the points of intersection of the two circles are $(-4, -2)$ and $(1, 3)$.

You can check that both points satisfy both original equations.

The linear equation $y = x + 2$ that appears in the solution to Example 9 is the equation of the line that passes through the two points of intersection $(-4, -2)$ and $(1, 3)$.

Activity 15 Finding the points of intersection of two circles

Find the points of intersection (if there are any) of the circles with equations

$$(x + 1)^2 + (y - 2)^2 = 9 \quad \text{and} \quad (x - 4)^2 + (y + 3)^2 = 36.$$

3.3 Using the computer for coordinate geometry

In this subsection you will use a computer to plot circles and calculate points of intersection.

Activity 16 Plotting circles using a computer

Work through Section 7 of the *Computer algebra guide*.



Activity 17 Using a computer to calculate points of intersection

- Use a computer to plot the parabola $y = 2x^2 - 10x + 14$ and the circle $x^2 - 6x + y^2 - 8y + 22 = 0$ on the same graph.
- By using the computer to solve the equations of the two curves in part (a) simultaneously (you saw how to do this in your study of Section 3 of the *Computer algebra guide*), calculate the coordinates of the points at which the two curves intersect, to two decimal places.
- What do you think is the maximum number of times that a parabola can intersect a circle?



4 Working in three dimensions

All the graphs and geometric diagrams that you have seen in this module up till now have been two-dimensional. As you know, the position of a point in two dimensions can be specified by using two coordinates, usually written as (x, y) . However the real world is three-dimensional. In this section you will see how to specify the positions of points in three-dimensional space.

4.1 Three-dimensional coordinates

To allow us to specify the positions of points in three-dimensional space, we extend the usual two-dimensional Cartesian coordinate system, by introducing a third axis, the z -axis, at right angles to the x - and y -axes. This is illustrated in Figure 20, which shows two different views of the same system of axes. Different views of the axes can be useful because some diagrams are clearer in one view than in another. Of course, it is impossible to include a truly three-dimensional diagram within this page of text: the figure is actually a two-dimensional impression of a three-dimensional image! Although two-dimensional coordinate systems are usually drawn with the positive direction of the y -axis pointing up, three-dimensional systems are usually drawn with the positive direction of the z -axis pointing up.

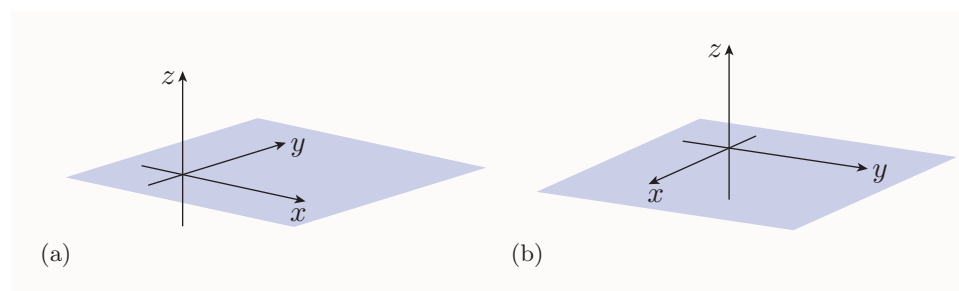


Figure 20 Two views of the same three-dimensional Cartesian coordinate system, with the x, y -plane shaded

In Figure 20 the plane that contains the x - and y -axes has been shaded. This plane is known as the x, y -plane. Similarly, the plane that contains the x - and z -axes is the x, z -plane, and the plane that contains the y - and z -axes is the y, z -plane. These three planes are known as the **coordinate planes**. The positive direction of the z -axis is called the **positive z -direction**, and similar terminology is used for the x - and y -axes.

There are in fact two different ways to introduce a z -axis perpendicular to both the x - and y -axes. The alternative way is for the positive direction of the z -axis to point down instead of up, as shown in Figure 21(a). If this system is rotated so that the z -axis points up, as shown in Figure 21(b), then the positions of the x - and y -axes are interchanged compared to those in Figure 20.

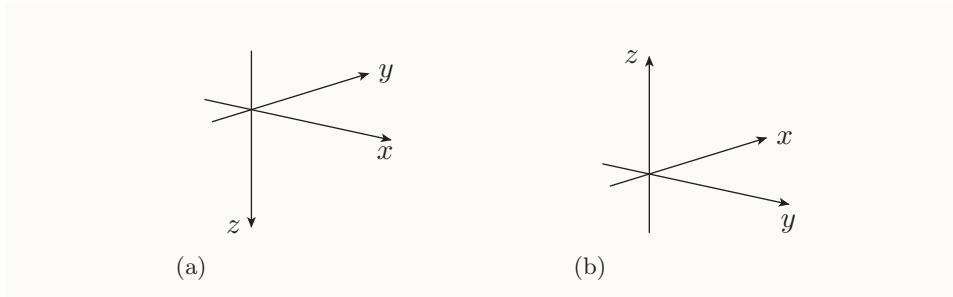


Figure 21 Two views of the alternative three-dimensional Cartesian coordinate system

The two different versions of a three-dimensional coordinate system are known as the **right-handed coordinate system** (Figure 20) and the **left-handed coordinate system** (Figure 21). One way of remembering which is which is to use the **right-hand rule**, which is illustrated in Figure 22.

You hold your right hand with the middle finger, first finger and thumb placed (roughly) perpendicular to each other, and the other two fingers closed. If your thumb and first finger point in the positive x - and y -directions, respectively, then your middle finger points in the positive z -direction of a right-handed coordinate system.

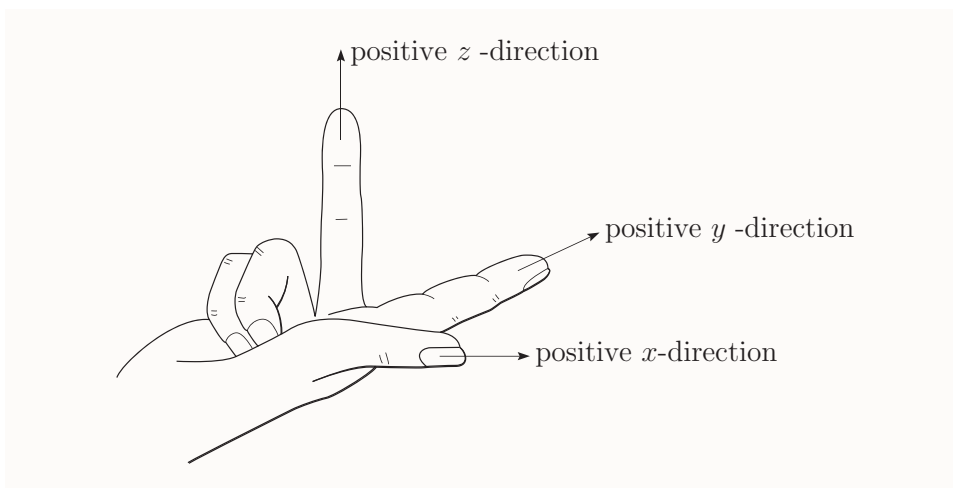


Figure 22 The right-hand rule

If you do the same with your left hand, then your middle finger points in the positive z -direction of a left-handed coordinate system.

Another way to remember which system is which is to use the **right-hand grip rule**, which is illustrated in Figure 23. If you hold your right hand in a fist, so that your fingers rotate from the positive part of the x -axis towards the positive part of the y -axis, then your thumb points in the positive z -direction of a right-handed coordinate system. Similarly, if you use your left hand, then your thumb points in the positive z -direction of a left-handed coordinate system.

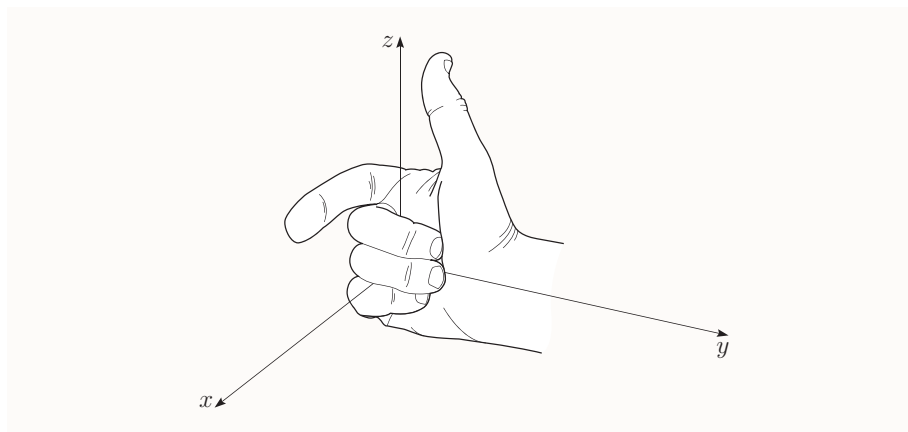


Figure 23 The right-hand grip rule

The right-handed coordinate system is usually used in preference to the left-handed one. In the rest of this module, wherever we use a three-dimensional coordinate system, we will assume that it is right-handed. Some mathematical formulas change slightly if a left-handed coordinate system is used, though none of the formulas in this module are affected by this.

The position of a point in three dimensions can be specified by three coordinates x , y and z , which are usually written as the triple (x, y, z) . The first two numbers in the triple are the coordinates of the point in the x, y -plane that lies directly below or above the original point. The third number specifies how far above or below the x, y -plane the original point lies. For example, the point with coordinates $(3, 4, 2)$ is illustrated in Figure 24.

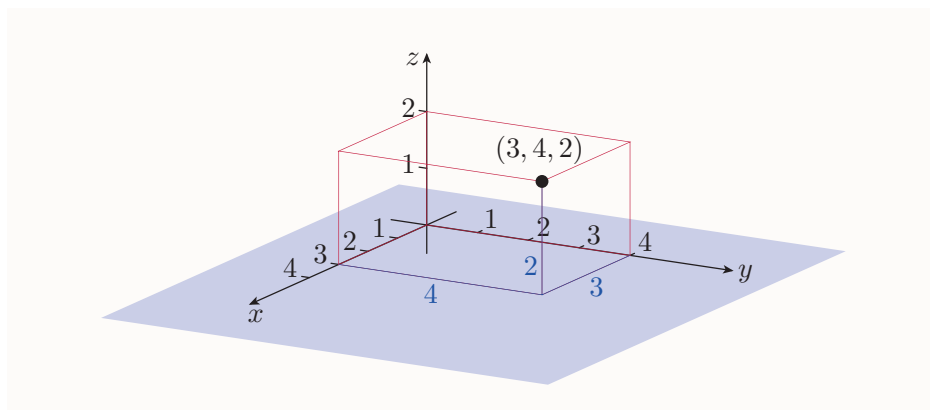
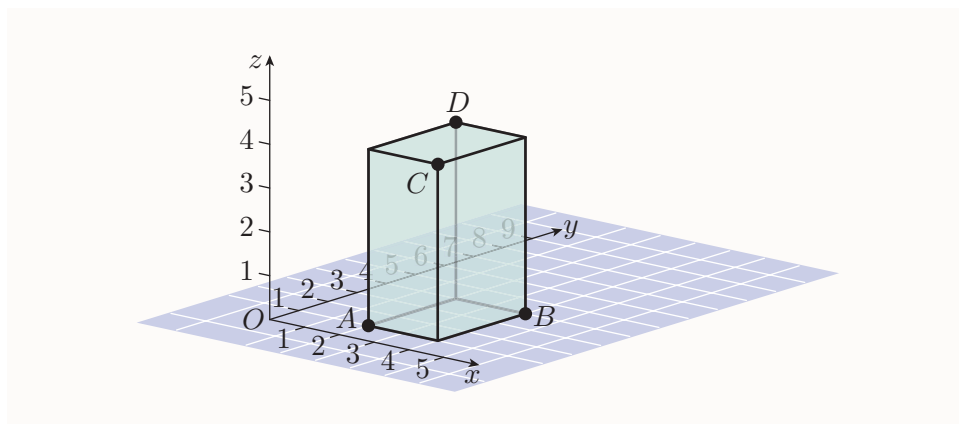


Figure 24 The point $(3, 4, 2)$; the x, y -plane is shaded

Activity 18 *Using three-dimensional coordinates*

The figure below shows a cuboid whose faces are parallel to the coordinate planes. Its bottom face lies in the x, y -plane, and its top face is 4 units above the x, y -plane.

What are the coordinates of the corners of the cuboid labelled A , B , C and D ?



4.2 The distance between two points in three dimensions

In this subsection you will meet a formula for the distance between any two points in three-dimensional space, in terms of their coordinates. It's similar to the formula that you met earlier for the distance between any two points in two dimensions.

Consider two points, $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$. For simplicity, let's consider the case in which $x_2 > x_1$, $y_2 > y_1$ and $z_2 > z_1$, as illustrated in Figure 25.

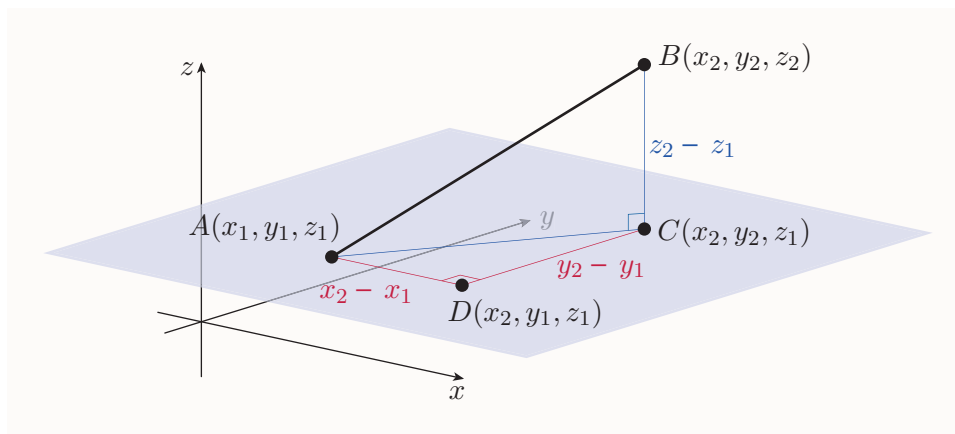


Figure 25 Two points in three dimensions

In the figure, the horizontal plane that contains the point A has been shaded. Let C be the point in this plane that lies directly below B , and let D be the point in the shaded plane such that AD is parallel to the x -axis and CD is parallel to the y -axis, as shown.

The distance AB is the length of the hypotenuse of the right-angled triangle with vertices A , B and C . So, by Pythagoras' theorem,

$$AB^2 = AC^2 + BC^2. \quad (6)$$

However the length AC that appears in this equation is itself the length of the hypotenuse of the right-angled triangle with vertices A , C and D . So, again by Pythagoras' theorem,

$$AC^2 = AD^2 + DC^2.$$

Using this equation to substitute for AC^2 in equation (6) gives

$$AB^2 = AD^2 + DC^2 + BC^2.$$

Since $AD = x_2 - x_1$, $DC = y_2 - y_1$ and $BC = z_2 - z_1$, this equation is

$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

In fact this equation holds *no matter where* the points A and B lie, for similar reasons to those that you saw for the distance formula in two dimensions. So we have the following result.

Distance formula (three dimensions)

The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Activity 19 *Finding distances between points in three dimensions*

Find the distance between the two points in each of the following pairs.

- (a) $(1, 3, 5)$ and $(4, 1, 7)$ (b) $(-1, 2, 5)$ and $(4, -1, 3)$

4.3 The equation of a sphere

In Section 2 you saw that a circle is the locus of points in the plane that are equidistant from a particular point. In three dimensions, the locus of points equidistant from a particular point is a sphere. The particular point is the centre of the sphere, and the distance from the centre to each point on the sphere is the radius of the sphere.

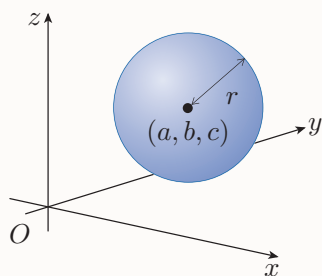


Figure 26 A sphere with centre (a, b, c) and radius r

If a sphere has centre (a, b, c) , and (x, y, z) is any point on the sphere, then the distance between these two points is equal to the radius of the sphere. That is,

$$\sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} = r.$$

Squaring both sides of this equation gives the following fact.

The standard form of the equation of a sphere

The sphere with centre (a, b, c) and radius r has equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

Activity 20 *Writing down the equations of spheres*

Write down the standard form of the equation of each of the spheres below.

- (a) The sphere with centre $(3, 5, 2)$ and radius 2
 (b) The sphere with centre $(2, -1, -3)$ and radius $\sqrt{3}$

Activity 21 *Finding the centres and radii of spheres*

Write down the centre and radius of the sphere specified by each of the following equations.

(a) $(x - 2)^2 + (y - 3)^2 + (z - 5)^2 = 16$

(b) $(x + 2)^2 + (y - 4)^2 + (z + 1)^2 = 7$

As with the equations of circles, the equations of spheres can be written in forms other than the standard form. For example, multiplying out the equation

$$(x + 1)^2 + (y - 3)^2 + (z - 2)^2 = 2$$

gives

$$x^2 + 2x + 1 + y^2 - 6y + 9 + z^2 - 4z + 4 = 2,$$

which simplifies to

$$x^2 + y^2 + z^2 + 2x - 6y - 4z + 12 = 0.$$

Each of the three equations above represents the same sphere.

If you have any equation for a sphere, then you can find its centre and radius by rearranging the equation into the standard form $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$. To do this, you use the same procedure as you used to rearrange the equations of circles into standard form, except that as well as completing the square in the subexpression formed by the terms in x^2 and x , and in the subexpression formed by the terms in y^2 and y , you have to complete the square in the subexpression formed by the terms in z^2 and z .

Not every equation with terms in x^2 , y^2 , z^2 , x , y , z and a constant term represents a sphere, because not every equation of this type can be rearranged into the form $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$. For example, if the coefficients of x^2 , y^2 and z^2 are not all the same, then the equation does not represent a sphere. Similarly, the equation $x^2 + y^2 + z^2 = -1$ does not represent a sphere, because no point (x, y, z) satisfies this equation.

Activity 22 *Identifying equations of spheres*

Which of the following equations represent a sphere? For each equation that represents a sphere, find the centre and radius of the sphere.

(a) $x^2 + y^2 + z^2 - 8x - 4y + 10z = -42$

(b) $x^2 - y^2 + 2z^2 + 2x + 4y - 4z - 2 = 0$

(c) $2x^2 + 2y^2 + 2z^2 + 4x - 12z + 12 = 0$

(d) $2x^2 + 2y^2 + 2z^2 + 8x - 4y + 13 = 0$

Non-spherical objects that are represented by equations with terms in x^2 , y^2 , z^2 , x , y , z and a constant term include *ellipsoids* and *hyperboloids*, as illustrated in Figure 27.

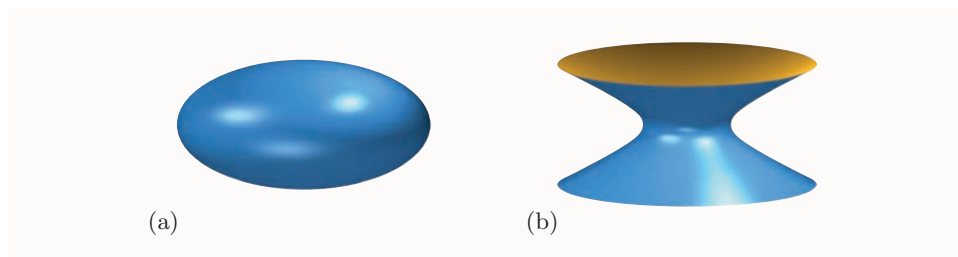


Figure 27 (a) The ellipsoid $x^2 + 2y^2 + 4z^2 = 1$ (b) the hyperboloid $x^2 + y^2 - 4z^2 = 1$

5 Vectors

This section marks the start of the second part of this unit, in which you'll learn about a type of mathematical object called a *vector*. Vectors play an important role in the study and analysis of phenomena in physics and engineering.

5.1 What is a vector?

Some mathematical quantities can't be specified just by stating their size – instead you need to state a size *and a direction*. For example, to fully describe the motion of a ship on the ocean at a point in time during its voyage, it's not enough to specify how fast the ship is moving – you also need to describe its direction of motion.

You saw other examples of this in Unit 2, when you considered objects moving along straight lines. To specify the position of a point P on a straight line relative to some other point, say O , on the line, you first choose one direction along the line to be the positive direction; then you state the distance between the two points, and attach a plus or minus sign to indicate the direction. The resulting quantity is called the *displacement* of P from O . For example, in Figure 28, if the positive direction is taken to be to the right, then the displacement of A from O is -2 cm, and the displacement of B from O is 4 cm.

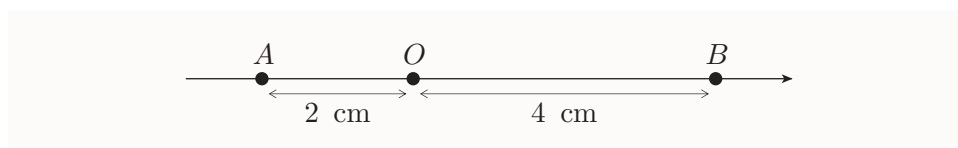
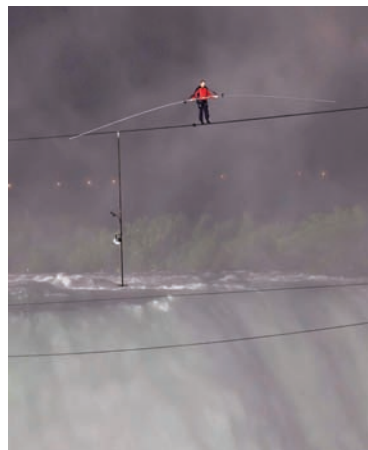


Figure 28 Positions along a straight line

Similarly, if an object is moving along a straight line, then you can describe its motion by giving its speed, and attaching a plus or minus sign



You can move only in one dimension along a tightrope (unless you fall off!)



You can move in two dimensions across a flat field



You can move in three dimensions in space

to indicate its direction. The resulting quantity is called the *velocity* of the object.

Plus and minus signs provide a convenient way to specify direction when you're dealing with movement along a straight line – that is, in one dimension. Examples of movement of this type include the motion of a car along a straight road, or that of a tightrope walker along a tightrope. However, we often need to deal with movement in two or three dimensions. For example, someone standing in a flat field can move across the field in two dimensions, and a person in space can move in three dimensions.

In general, **displacement** is the position of one point relative to another, whether in one, two or three dimensions. To specify a displacement, you need to give both a distance and a direction. For example, consider the points O , P and Q in Figure 29. You can specify the displacement of P from O by saying that it is 1 km north-west of O . Similarly, you can specify the displacement of Q from O by saying that it is 1 km north-east of O .



Figure 29 Points in a plane

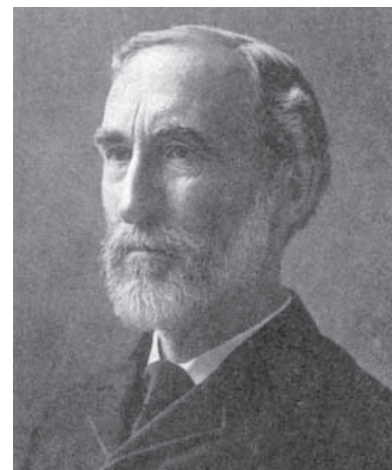
Just as distance together with direction is called *displacement*, so speed together with direction is called **velocity**. For example, if you say that someone is walking at a speed of 5 km h^{-1} south, then you're specifying a velocity.

Quantities, such as displacement and velocity, that have both a size and a direction are called **vectors**, or **vector quantities**. (In Latin, the word *vector* means 'carrier'.) Another example of a vector quantity is *force*. The size of a vector is usually called its **magnitude**.

In contrast to vectors, quantities that have size but no direction are called **scalars**, or **scalar quantities**. Examples of scalars include distance, speed, time, temperature and volume. So a scalar is a number, possibly with a unit.

Notice that the magnitude of the displacement of one point from another is the distance between the two points, and the magnitude of the velocity of an object is its speed. In everyday English the words 'speed' and 'velocity' are often used interchangeably, but in scientific and mathematical terminology there is an important difference: speed is a scalar and velocity is a vector.

The concept of vectors evolved over a long time. Isaac Newton (1642–1727) dealt extensively with vector quantities, but never formalised them. The first exposition of what we would today know as vectors was by Josiah Willard Gibbs in 1881, in his *Elements of vector analysis*. This work was derived from earlier ideas of William Rowan Hamilton (1805–1865).



Josiah Willard Gibbs
(1839–1903)



Any vector with non-zero magnitude can be represented by an *arrow*, which is a line segment with an associated direction, like the one in Figure 30. The length of the arrow represents the magnitude of the vector, according to some chosen scale, and the direction of the arrow represents the direction of the vector. Two-dimensional vectors are represented by arrows in a plane, and three-dimensional vectors are represented by arrows in three-dimensional space. For example, the arrow in Figure 30 might represent a displacement of 30 km north-west, if you're using a scale of 1 cm to represent 10 km. Alternatively, the same arrow might represent a velocity of 30 m s^{-1} north-west, if you're using a scale of 1 cm to represent 10 m s^{-1} .

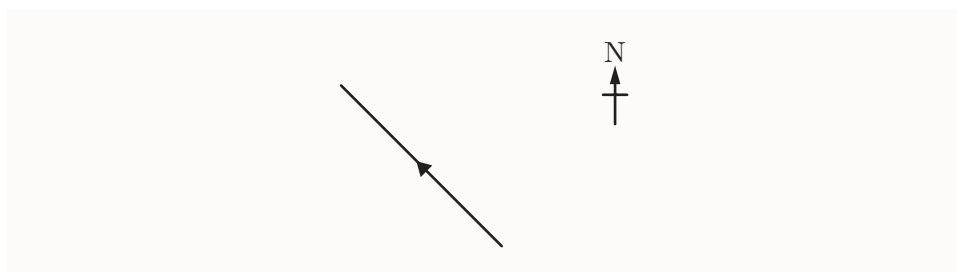


Figure 30 An arrow that represents a vector

Once you've chosen a scale, any two arrows with the same length and the same direction represent the *same* vector. For example, all the arrows in Figure 31 represent the same vector.

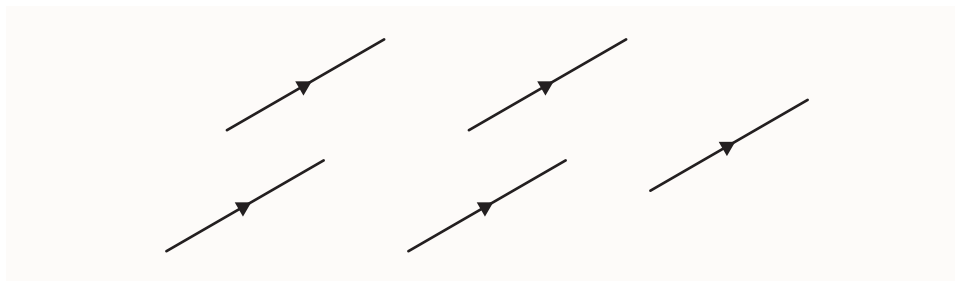


Figure 31 Several arrows representing the same vector

Vectors are often denoted by lower-case letters. We distinguish them from scalars by using a bold typeface in typed text, and by underlining them in handwritten text. For example, the vector in Figure 30 might be denoted by $\mathbf{v}$ in print, or handwritten as $\underline{v}$. These conventions prevent readers from confusing vector and scalar quantities.

Remember to underline handwritten vectors (and make typed ones bold) in your own work.

Vectors that represent displacements are sometimes called **displacement vectors**. There is a useful alternative notation for such vectors. If P and Q are any two points, then the vector that specifies the displacement from P to Q (illustrated in Figure 32) is denoted by $\overrightarrow{PQ}$.

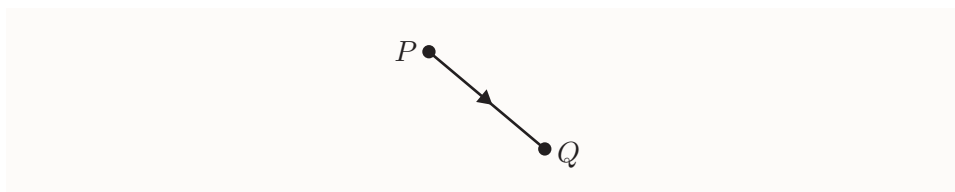


Figure 32 The vector $\overrightarrow{PQ}$

The magnitude of a vector $\mathbf{v}$ is a scalar quantity. It is denoted by $|\mathbf{v}|$, which is read as ‘the magnitude of $\mathbf{v}$ ’, ‘the modulus of $\mathbf{v}$ ’, or simply ‘mod $\mathbf{v}$ ’. For example, if the vector $\mathbf{v}$ represents a velocity of 30 m s^{-1} north-west, then $|\mathbf{v}| = 30 \text{ m s}^{-1}$. Similarly, if the vector $\overrightarrow{PQ}$ represents a displacement of 3 m south-east, then $|\overrightarrow{PQ}| = 3 \text{ m}$. Remember that the distance between the points P and Q can also be denoted by PQ , so

$$PQ = |\overrightarrow{PQ}|.$$

Notice that the notation for the magnitude of a vector is the same as the notation for the magnitude of a scalar that you met in Unit 3. For example, you saw there that $|-3| = 3$. So this notation can be applied to either vectors or scalars.

When we're working with vectors, it's often convenient, for simplicity, not to distinguish between vectors and the arrows that represent them. For example, we might say 'the vector shown in the diagram' rather than 'the vector represented by the arrow shown in the diagram'. This convention is used throughout the rest of this unit.

Over the next few pages you'll learn the basics of working with vectors.

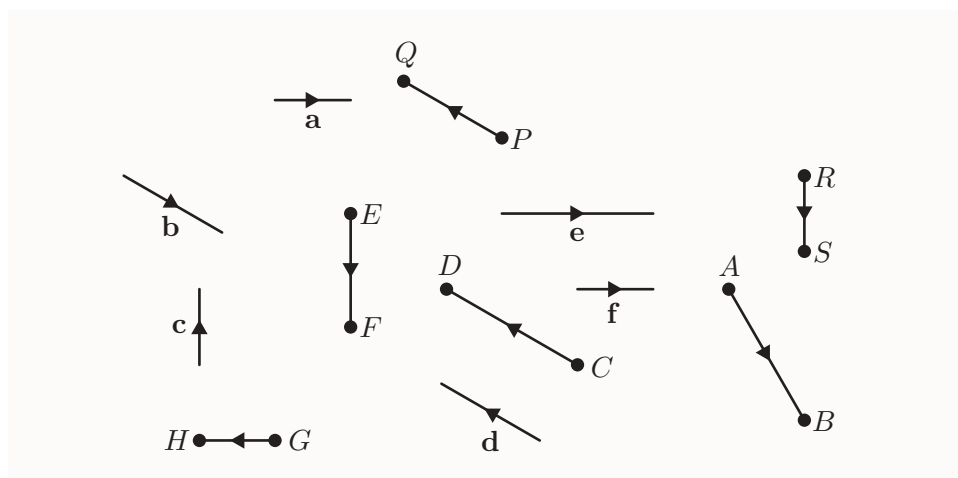
Equal vectors

As you'd expect, two vectors are **equal** if they have the same magnitude and the same direction.

Activity 23 Identifying equal vectors

The following diagram shows several displacement vectors.

- (a) Which vector is equal to the vector **a**?
- (b) Which vector is equal to the vector $\overrightarrow{PQ}$?



The zero vector

The *zero vector* is defined as follows.

Zero vector

The **zero vector**, denoted by **0** (bold zero), is the vector whose magnitude is zero. It has no direction.

The zero vector is handwritten as 0 (zero underlined).

For example, the displacement of a particular point from itself is the zero vector, as is the velocity of an object that is not moving.

Addition of vectors

To understand how to *add* two vectors, it's helpful to think about displacement vectors. For example, consider the situation shown in Figure 33. Suppose that an object is positioned at a point P and you first move it to the point Q , then you move it again to the point R . The two displacements are the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, respectively, and the overall, combined displacement is the vector $\overrightarrow{PR}$. This method of combining two vectors to produce another vector is called **vector addition**.

We write

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}.$$

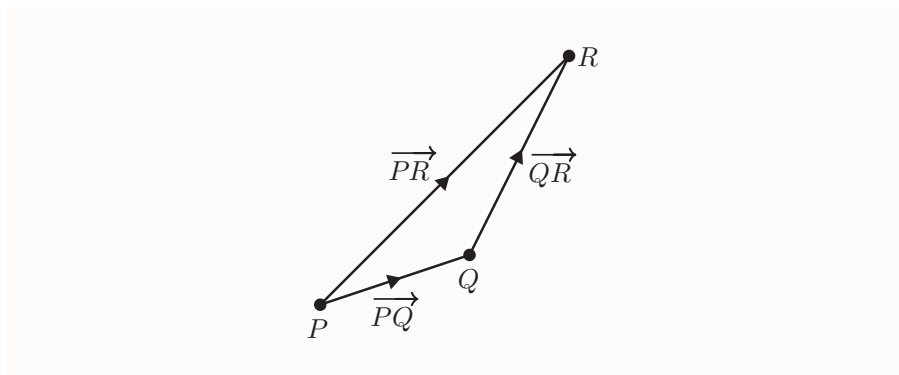
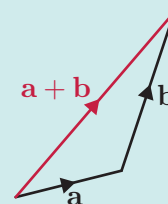


Figure 33 The result of adding two displacement vectors

Vectors are always added in this way. The general rule is called the *triangle law for vector addition*, and it can be stated as follows.

Triangle law for vector addition

To find the sum of two vectors $\mathbf{a}$ and $\mathbf{b}$, place the tail of $\mathbf{b}$ at the tip of $\mathbf{a}$. Then $\mathbf{a} + \mathbf{b}$ is the vector from the tail of $\mathbf{a}$ to the tip of $\mathbf{b}$.



The sum of two vectors is also called their **resultant** or **resultant vector**.

You can add two vectors in either order, and you get the same result either way. This is illustrated in Figure 34. Diagrams (a) and (b) show how the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{b} + \mathbf{a}$ are found using the triangle law for vector addition. When you place these two diagrams together, as shown in diagram (c), the two resultant vectors coincide, because they lie along the diagonal of the parallelogram formed by the two copies of $\mathbf{a}$ and $\mathbf{b}$.

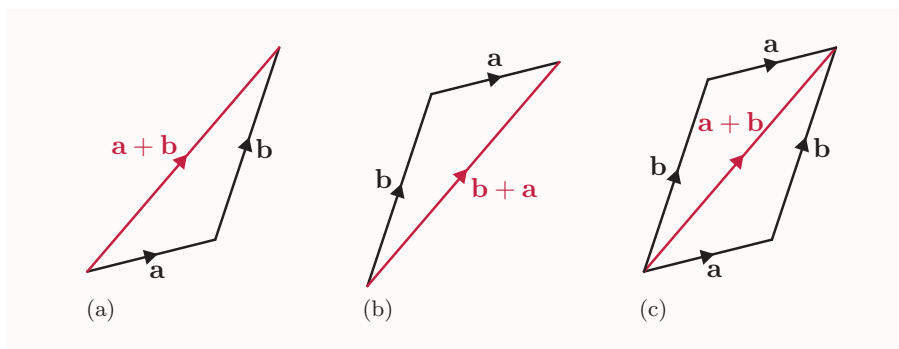
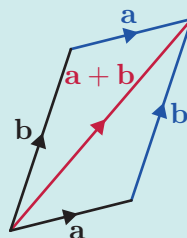


Figure 34 The vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{b} + \mathbf{a}$ are equal

In fact, Figure 34(c) gives an alternative way to add two vectors, the *parallelogram law for vector addition*, which can be stated as follows.

Parallelogram law for vector addition

To find the sum of two vectors $\mathbf{a}$ and $\mathbf{b}$, place their tails together, and complete the resulting figure to form a parallelogram. Then $\mathbf{a} + \mathbf{b}$ is the vector formed by the diagonal of the parallelogram, starting from the point where the tails of $\mathbf{a}$ and $\mathbf{b}$ meet.



The parallelogram law is convenient in some contexts, and you'll use it in Unit 12. In this unit we'll always use the triangle law, as it's simpler in the sorts of situations that we'll deal with here.

You can add more than two vectors together. To add several vectors, you place them all tip to tail, one after another; then their sum is the vector from the tail of the first vector to the tip of the last vector. For example, Figure 35 illustrates how three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are added.

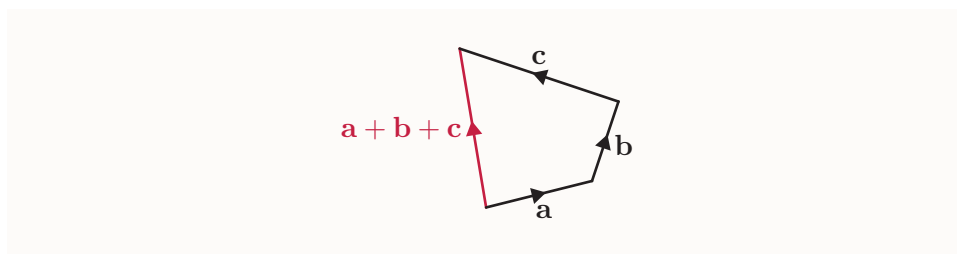
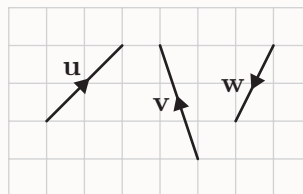


Figure 35 The sum of three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$

The order in which you add the vectors doesn't matter – you always get the same resultant vector.

Activity 24 Adding vectors

The diagram below shows three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ drawn on a grid.



Draw arrows representing the following vector sums. (Use squared paper or sketch a grid.)

- (a) $\mathbf{u} + \mathbf{v}$ (b) $\mathbf{u} + \mathbf{w}$ (c) $\mathbf{v} + \mathbf{w}$ (d) $\mathbf{u} + \mathbf{v} + \mathbf{w}$ (e) $\mathbf{u} + \mathbf{u}$

As you'd expect, adding the zero vector to any vector leaves it unchanged. That is, for any vector $\mathbf{a}$,

$$\mathbf{a} + \mathbf{0} = \mathbf{a}.$$

Note that you can't add a vector to a scalar. Expressions such as $\mathbf{v} + x$, where $\mathbf{v}$ is a vector and x is a scalar, are meaningless.

Negative of a vector

The *negative* of a vector $\mathbf{a}$ is denoted by $-\mathbf{a}$, and is defined as follows.

Negative of a vector

The **negative** of a vector $\mathbf{a}$, denoted by $-\mathbf{a}$, is the vector with the same magnitude as $\mathbf{a}$, but the opposite direction.



For any points P and Q , the position vectors $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ have the property that $-\overrightarrow{PQ} = \overrightarrow{QP}$, since $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ have opposite directions.

If you add any vector $\mathbf{a}$ to its negative $-\mathbf{a}$, by placing the two vectors tip to tail in the usual way, then you get the zero vector. In other words, for any vector $\mathbf{a}$,

$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0},$$

as you'd expect.

The negative of the zero vector is the zero vector; that is, $-\mathbf{0} = \mathbf{0}$.

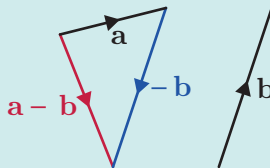
Subtraction of vectors

To see how vector *subtraction* is defined, first consider the subtraction of numbers. Subtracting a number is the same as adding the negative of the number. In other words, if a and b are numbers, then $a - b$ means the same as $a + (-b)$. We use this idea to define vector subtraction, as follows.

Vector subtraction

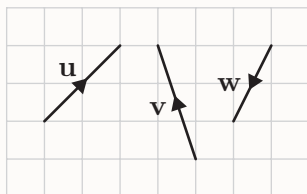
To subtract $\mathbf{b}$ from $\mathbf{a}$, add $-\mathbf{b}$ to $\mathbf{a}$.
That is,

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$



Activity 25 Subtracting vectors

The diagram below shows three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ drawn on a grid.



Draw arrows representing the following negatives and differences of vectors. (Use squared paper or sketch a grid.)

- (a) $-\mathbf{v}$ (b) $-\mathbf{w}$ (c) $\mathbf{u} - \mathbf{v}$ (d) $\mathbf{v} - \mathbf{w}$ (e) $\mathbf{u} + \mathbf{v} - \mathbf{w}$

You have already seen that for any vector $\mathbf{a}$, we have $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$. That is, for any vector $\mathbf{a}$, we have $\mathbf{a} - \mathbf{a} = \mathbf{0}$, as you would expect.

Multiplication of vectors by scalars

You can multiply vectors by scalars. To understand what this means, first consider the effect of adding a vector $\mathbf{a}$ to itself, as illustrated in Figure 36.

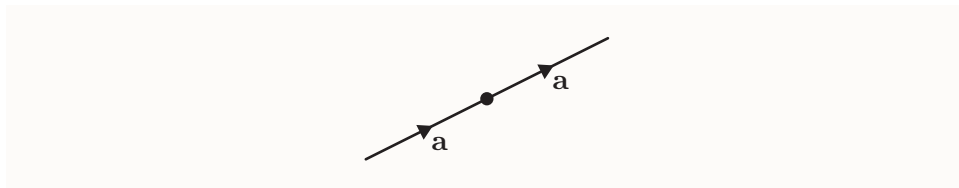


Figure 36 A vector $\mathbf{a}$ added to itself

The resultant vector $\mathbf{a} + \mathbf{a}$ has the same direction as $\mathbf{a}$, but twice the magnitude. We denote it by $2\mathbf{a}$. We say that this vector is a **scalar multiple** of the vector $\mathbf{a}$, since 2 is a scalar quantity. In general, scalar multiplication of vectors is defined as below. Note that in this box the notation $|m|$ means the magnitude of the *scalar* m .

Scalar multiple of a vector

Suppose that $\mathbf{a}$ is a vector. Then, for any non-zero real number m , the **scalar multiple** $m\mathbf{a}$ of $\mathbf{a}$ is the vector

- whose magnitude is $|m|$ times the magnitude of $\mathbf{a}$
- that has the same direction as $\mathbf{a}$ if m is positive, and the opposite direction if m is negative.

Also, $0\mathbf{a} = \mathbf{0}$.

(That is, the number zero times the vector $\mathbf{a}$ is the zero vector.)

Remember that a scalar multiple of a vector is a *vector*.

Various scalar multiples of a vector $\mathbf{a}$ are shown in Figure 37.

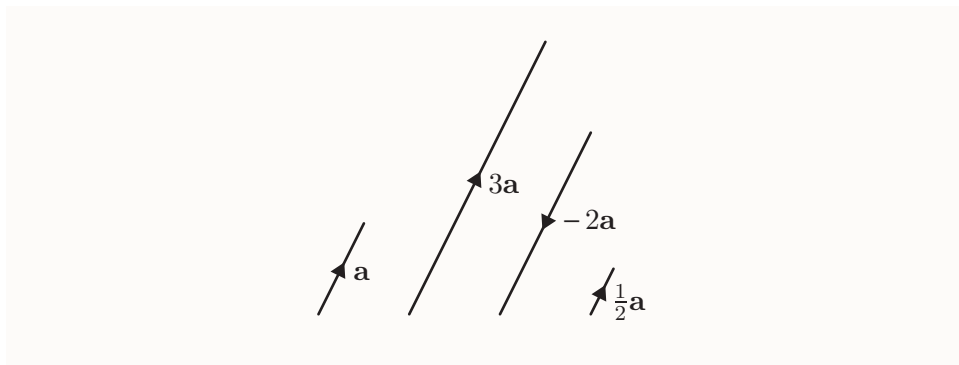


Figure 37 Scalar multiples of a vector $\mathbf{a}$

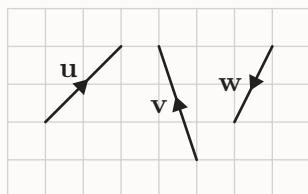
By the definition above, if $\mathbf{a}$ is any vector, then $(-1)\mathbf{a}$ is the vector with the same magnitude as $\mathbf{a}$ but the opposite direction. In other words, as

you would expect,

$$(-1)\mathbf{a} = -\mathbf{a}.$$

Activity 26 *Multiplying vectors by scalars*

The diagram below shows three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ drawn on a grid.



Draw arrows representing the following vectors. (Use squared paper or sketch a grid.)

- (a) $3\mathbf{u}$ (b) $-2\mathbf{v}$ (c) $\frac{1}{2}\mathbf{v}$ (d) $3\mathbf{u} - 2\mathbf{v}$ (e) $-2\mathbf{v} + \mathbf{w}$

The next example illustrates how you can use scalar multiples of vectors to represent quantities in practical situations.

Example 10 *Scaling velocities*

Suppose that the vector $\mathbf{u}$ represents the velocity of a car travelling with speed 50 km h^{-1} along a straight road heading north. Write down, in terms of $\mathbf{u}$, the velocity of a second car that is travelling in the same direction as the first with a speed of 75 km h^{-1} .

Solution

The velocity of the second car has the same direction as $\mathbf{u}$, and hence is a scalar multiple of it. The speed of the second car is 1.5 times the speed of the first.

The velocity of the second car is $1.5\mathbf{u}$.

The activity below involves winds measured in *knots*. A knot is a unit of speed often used in meteorology, and in air and maritime navigation. Its usual abbreviation is kn, and $1 \text{ kn} = 1.852 \text{ km h}^{-1}$.

Conventionally, the direction of a wind is usually given as the direction *from* which it blows, rather than the direction that it blows towards. So, for example, a southerly wind is one blowing from the south, towards the north.

Activity 27 *Scaling velocities*

Suppose that the vector $\mathbf{v}$ represents the velocity of a wind of 35 knots blowing from the north-east. Express the following vectors in terms of $\mathbf{v}$.

- (a) The velocity of a wind of 70 knots blowing from the north-east.
- (b) The velocity of a wind of 35 knots blowing from the south-west.

5.2 Vector algebra

In Subsection 5.1, you met some properties of the addition, subtraction and scalar multiplication of vectors. For example, you saw that for any vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}, \quad \mathbf{a} + \mathbf{0} = \mathbf{a}, \quad \mathbf{a} - \mathbf{a} = \mathbf{0} \quad \text{and} \quad \mathbf{a} + \mathbf{a} = 2\mathbf{a}.$$

All the properties that you met can be deduced from the eight basic algebraic properties of vectors listed below.

Properties of vector algebra

The following properties hold for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and all scalars m and n .

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}$
6. $(m + n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$
7. $m(n\mathbf{a}) = (mn)\mathbf{a}$
8. $1\mathbf{a} = \mathbf{a}$

These properties are similar to properties that hold for addition, subtraction and multiplication of real numbers. Similar properties also hold for many different systems of mathematical objects. You'll meet further examples of such systems later in the module.

Property 1 says that the order in which two vectors are added does not matter. This property can be described by saying that vector addition is **commutative**. Similarly, addition of real numbers is commutative, because $a + b = b + a$ for all real numbers a and b , and multiplication of real numbers is commutative, because $ab = ba$ for all real numbers a and b . Subtraction of real numbers is not commutative, because it is not true that $a - b = b - a$ for all real numbers a and b .

Property 2 says that finding $\mathbf{a} + \mathbf{b}$ and then adding $\mathbf{c}$ to the result gives the same final answer as finding $\mathbf{b} + \mathbf{c}$ and then adding $\mathbf{a}$ to the result. You might like to check this for a particular case, by drawing the vectors as arrows. This property is described by saying that vector addition is **associative**. It allows us to write the expression $\mathbf{a} + \mathbf{b} + \mathbf{c}$ without there being any ambiguity in what is meant – you can interpret it as either $(\mathbf{a} + \mathbf{b}) + \mathbf{c}$ or $\mathbf{a} + (\mathbf{b} + \mathbf{c})$, because both mean the same. Addition and multiplication of real numbers are also associative operations.

Property 5 says that adding two vectors and then multiplying the result by a scalar gives the same final answer as multiplying each of the two vectors individually by the scalar and then adding the two resulting vectors. This property is described by saying that scalar multiplication is **distributive** over the addition of vectors.

Similarly, property 6 says that scalar multiplication is distributive over the addition of scalars.

You will notice that nothing has been said about whether vectors can be multiplied or divided by other vectors. There is a useful way to define multiplication of two vectors – two different ways, in fact! You will meet one of these ways in Section 7. Division of a vector by another vector is not possible.

The properties in the box above allow you to perform some operations on vector expressions in a similar way to real numbers, as illustrated in the following example.

Example 11 *Simplifying a vector expression*

Simplify the vector expression

$$2(\mathbf{a} + \mathbf{b}) + 3(\mathbf{b} + \mathbf{c}) - 5(\mathbf{a} + \mathbf{b} - \mathbf{c}).$$

Solution

 Expand the brackets, using property 5 above. 

$$\begin{aligned} & 2(\mathbf{a} + \mathbf{b}) + 3(\mathbf{b} + \mathbf{c}) - 5(\mathbf{a} + \mathbf{b} - \mathbf{c}) \\ &= 2\mathbf{a} + 2\mathbf{b} + 3\mathbf{b} + 3\mathbf{c} - 5\mathbf{a} - 5\mathbf{b} + 5\mathbf{c} \end{aligned}$$

 Collect like terms, using property 6 above. 

$$\begin{aligned} &= 2\mathbf{a} - 5\mathbf{a} + 2\mathbf{b} + 3\mathbf{b} - 5\mathbf{b} + 3\mathbf{c} + 5\mathbf{c} \\ &= 8\mathbf{c} - 3\mathbf{a}. \end{aligned}$$

The properties in the box above also allow you to manipulate equations containing vectors, which are known as **vector equations**, in a similar way to ordinary equations. For example, you can add or subtract vectors on both sides of such an equation, and you can multiply or divide both

sides by a non-zero scalar. You can use these methods to rearrange a vector equation to make a particular vector the subject, or to solve a vector equation for an unknown vector.

Activity 28 Manipulating vector expressions and equations

- (a) Simplify the vector expression $4(\mathbf{a} - \mathbf{c}) + 3(\mathbf{c} - \mathbf{b}) + 2(2\mathbf{a} - \mathbf{b} - 3\mathbf{c})$.
- (b) Rearrange each of the following vector equations to express $\mathbf{x}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
- (i) $2\mathbf{b} + 4\mathbf{x} = 7\mathbf{a}$ (ii) $3(\mathbf{b} - \mathbf{a}) + 5\mathbf{x} = 2(\mathbf{a} - \mathbf{b})$

5.3 Using vectors

In this subsection you'll see some examples of how you can use two-dimensional vectors in practical situations.

When you use a vector to represent a real-world quantity, you need a means of expressing its direction. For a two-dimensional vector, one way to do this is to state the angle measured from some chosen reference direction to the direction of the vector. You have to make it clear whether the angle is measured clockwise or anticlockwise.

If the vector represents the displacement or velocity of an object such as a ship or an aircraft, then its direction is often given as a *compass bearing*. There are various different types of compass bearings, but in this module we will use the following type.

A **bearing** is an angle between 0° and 360° , measured clockwise in degrees from north to the direction of interest.

For example, Figure 38 shows a vector $\mathbf{v}$ with a bearing of 150° .



A navigational compass

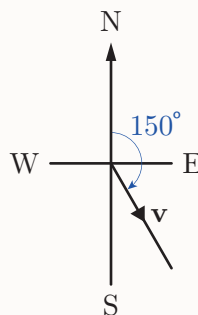


Figure 38 A vector with a bearing of 150°

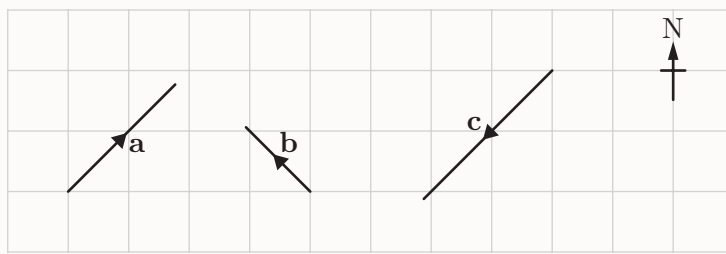
When bearings are used in practice, there are various possibilities for the meaning of ‘north’. It can mean magnetic north (the direction in which a compass points), true north (the direction to the North Pole) or grid north (the direction marked as north on a particular map). We’ll assume that one of these has been chosen in any particular situation.

Notice that the rotational direction in which bearings are measured is *opposite* to that in which angles are usually measured in mathematics.

Bearings are measured *clockwise* (from north), whereas in Unit 4 you saw angles measured *anticlockwise* (from the positive direction of the x -axis).

Activity 29 Working with bearings

- (a) Write down the bearings that specify the directions of the following vectors. (The acute angle between each vector and the gridlines is 45° .)



- (b) Draw arrows to represent vectors (of any magnitude) with directions given by the following bearings.
- (i) 90° (ii) 135° (iii) 270°

When you work with the directions of vectors expressed using angles, you often need to use trigonometry, as illustrated in the next example.

**Example 12** Adding two perpendicular vectors

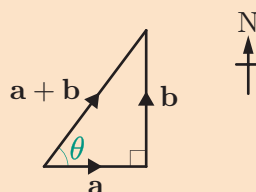
An explorer walks for 3 km on a bearing of 90° , then turns and walks for 4 km on a bearing of 0° .

Find the magnitude and bearing of his resultant displacement, giving the bearing to the nearest degree.

Solution

Represent the first part of the walk by the vector $\mathbf{a}$, and the second part by the vector $\mathbf{b}$. Then the resultant displacement is $\mathbf{a} + \mathbf{b}$.

Draw a diagram showing $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} + \mathbf{b}$. Since $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, you obtain a right-angled triangle.



Use Pythagoras' theorem to find the magnitude of $\mathbf{a} + \mathbf{b}$.

Since $|\mathbf{a}| = 3$ km, $|\mathbf{b}| = 4$ km and the triangle is right-angled,

$$|\mathbf{a} + \mathbf{b}| = \sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \text{ km.}$$

Use basic trigonometry to find one of the acute angles in the triangle.

From the diagram,

$$\tan \theta = \frac{|\mathbf{b}|}{|\mathbf{a}|} = \frac{4}{3},$$

$$\text{so } \theta = \tan^{-1}\left(\frac{4}{3}\right) = 53^\circ \text{ (to the nearest degree).}$$

Hence find the bearing of $\mathbf{a} + \mathbf{b}$.

The bearing of $\mathbf{a} + \mathbf{b}$ is $90^\circ - \theta = 37^\circ$ (to the nearest degree).

State a conclusion, remembering to include units.

So the resultant displacement has magnitude 5 km and a bearing of approximately 37° .

Activity 30 *Adding two perpendicular vectors*

A yacht sails on a bearing of 60° for 5.3 km, then turns through 90° and sails on a bearing of 150° for a further 2.1 km.

Find the magnitude and bearing of the yacht's resultant displacement. Give the magnitude of the displacement in km to one decimal place, and the bearing to the nearest degree.

The vectors that were added in Example 12 and Activity 30 were perpendicular, so only basic trigonometry was needed. In the next activity, you're asked to add two displacement vectors that aren't perpendicular. You need to draw a clear diagram and use the sine and cosine rules to find the required lengths and angles.

Activity 31 *Adding two non-perpendicular vectors*

The grab of a robotic arm moves 40 cm from its starting point on a bearing of 90° to pick up an object, and then moves the object 20 cm on a bearing of 315° .

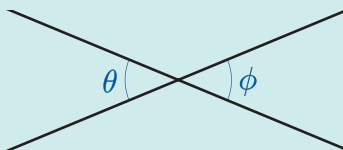
Find the resultant displacement of the grab, giving the magnitude to the nearest centimetre, and the bearing to the nearest degree.

In some examples involving vectors, it can be quite complicated to work out the angles that you need to know from the information that you have. You often need to use the following geometric properties.

Opposite, corresponding and alternate angles

Where two lines intersect:

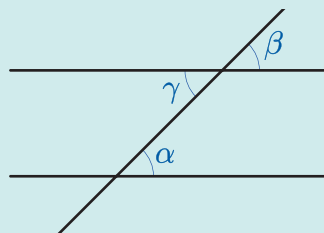
opposite angles are equal
(for example, $\theta = \phi$).

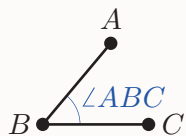


Where a line intersects parallel lines:

corresponding angles are equal
(for example, $\alpha = \beta$);

alternate angles are equal
(for example, $\alpha = \gamma$).



Figure 39 $\angle ABC$ 

The next example illustrates how to use some of these geometric properties. You should find the tutorial clip for this example particularly helpful.

The example uses the standard notation $\angle ABC$ (read as ‘angle ABC ’) for the acute angle at the point B between the line segments AB and BC . This is illustrated in Figure 39.

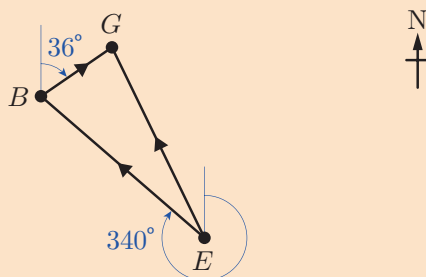
Example 13 Adding two non-perpendicular vectors

The displacement from Exeter to Belfast is 460 km with a bearing of 340° , and the displacement from Belfast to Glasgow is 173 km with a bearing of 36° . Use this information to find the magnitude (to the nearest kilometre) and direction (as a bearing, to the nearest degree) of the displacement from Exeter to Glasgow.

Solution

Denote Exeter by E , Belfast by B , and Glasgow by G .

Draw a diagram showing the displacement vectors $\vec{EB}$, $\vec{BG}$ and their resultant $\vec{EG}$. Mark the angles that you know. Mark or state any magnitudes that you know.



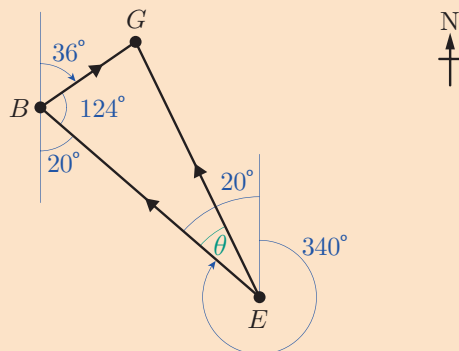
We know that $EB = 460$ km and $BG = 173$ km.

To enable you to calculate the magnitude and bearing of $\vec{EG}$, you need to find an angle in triangle BEG . Use geometric properties to find $\angle EBG$.

Since the bearing of $\vec{EB}$ is 340° , the acute angle at E between $\vec{EB}$ and north is $360^\circ - 340^\circ = 20^\circ$, as shown in the diagram below.

Hence, since alternate angles are equal, the acute angle at B between $\vec{EB}$ and south is also 20° .

So $\angle EBG = 180^\circ - 36^\circ - 20^\circ = 124^\circ$.



Now use the cosine rule to calculate EG .

Applying the cosine rule in triangle EBG gives

$$EG^2 = EB^2 + BG^2 - 2 \times EB \times BG \times \cos 124^\circ,$$

so

$$\begin{aligned} EG &= \sqrt{460^2 + 173^2 - 2 \times 460 \times 173 \times \cos 124^\circ} \\ &= 574.91 \dots = 575 \text{ km (to the nearest km).} \end{aligned}$$

To find the bearing of $\overrightarrow{EG}$, first find $\angle BEG$.

Let $\angle BEG = \theta$, as marked in the diagram. Then, by the sine rule,

$$\frac{BG}{\sin \theta} = \frac{EG}{\sin 124^\circ}$$

$$\begin{aligned} \sin \theta &= \frac{BG \sin 124^\circ}{EG} \\ &= \frac{173 \sin 124^\circ}{574.91 \dots}. \end{aligned}$$

Now

$$\sin^{-1} \left(\frac{173 \sin 124^\circ}{574.91 \dots} \right) = 14.44 \dots^\circ,$$

so

$$\theta = 14.44 \dots^\circ \quad \text{or} \quad \theta = 180^\circ - 14.44 \dots^\circ = 165.55 \dots^\circ.$$

If $\theta = 165.55 \dots^\circ$, then the sum of θ and $\angle EBG$ (two of the angles in triangle EBG) is greater than 180° , which is impossible. So $\theta = 14.44 \dots^\circ$.

Hence the bearing of $\overrightarrow{EG}$ is

$$340^\circ + \theta = 340^\circ + 14.44 \dots^\circ = 354.44 \dots^\circ.$$

State a conclusion.

The displacement of Glasgow from Exeter is 575 km (to the nearest km) on a bearing of 354° (to the nearest degree).

As mentioned in Subsection 5.1, velocity is a vector quantity, since it is the speed with which an object is moving together with its direction of motion. So the methods that you have seen for adding displacements can also be applied to velocities.

It may at first seem strange to add velocities, but consider the following situation. Suppose that a boy is running across the deck of a ship. If the ship is motionless in a harbour, then the boy's velocity relative to the sea bed is the same as his velocity relative to the ship.

However, if the ship is moving, then the boy's velocity relative to the sea bed is a combination of his velocity relative to the ship and the ship's velocity relative to the sea bed. In fact, the boy's resultant velocity relative to the sea bed is the vector sum of the two individual velocities.



Activity 32 Adding velocities

A ship is steaming at a speed of 10.0 m s^{-1} on a bearing of 30° in still water. A boy runs across the deck of the ship from the port side to the starboard side, perpendicular to the direction of motion of the ship, with a speed of 4.0 m s^{-1} relative to the ship. (The port and starboard sides of a ship are the sides on the left and right, respectively, of a person on board facing the front.)

Find the resultant velocity of the boy, giving the speed in m s^{-1} to one decimal place and the bearing to the nearest degree.

When a ship sails in a current, or an aircraft flies through a wind, its actual velocity is the resultant of the velocity that it would have if the water or air were still, and the velocity of the current or wind. In particular, the direction in which the ship or aircraft is *pointing* – this is called its **heading**, when it is given as a bearing – may be different from the direction in which it is actually moving, which is called its **course**. This is because the current or wind may cause it to continuously drift to one side.

Activity 33 Finding the resultant velocity of a ship in a current

A ship has a speed in still water of 5.7 m s^{-1} and is sailing on a heading of 230° . However, there is a current in the water of speed 2.5 m s^{-1} flowing on a bearing of 330° . Find the resultant velocity of the ship, giving the speed in m s^{-1} to one decimal place and the bearing to the nearest degree.

6 Component form of a vector

In Section 5, you saw how to specify two-dimensional vectors by stating their magnitudes and directions, and how to work with them using trigonometry. The calculations involved can be quite complicated, especially when you want to find a sum of more than two vectors. Three-dimensional vectors can be dealt with in a similar way, though the calculations are usually even more complicated. Fortunately, there is an alternative way to represent and manipulate vectors, which makes them easier to work with. This involves expressing them in terms of *components*.

6.1 Representing vectors using components

To express a vector in terms of components, you first introduce a system of coordinate axes with which to work, as shown in Figure 40. You can choose any perpendicular directions for the axes, but usually you choose them to be horizontal and vertical, or you align them with directions intrinsic to the situation that you're dealing with. You label the axes x and y (in two dimensions), or x , y and z (in three dimensions), in the usual way.

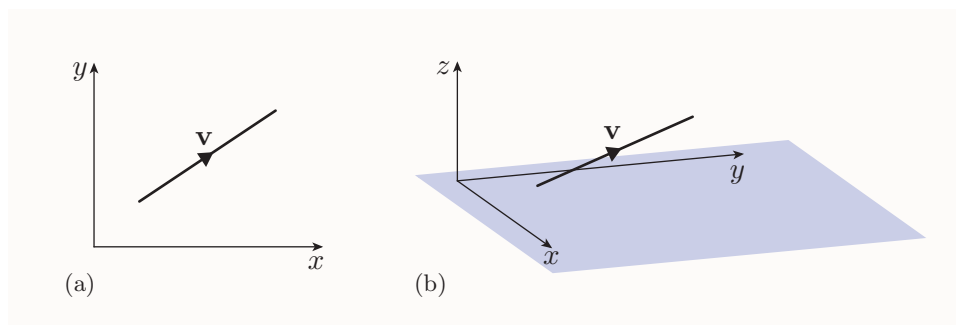


Figure 40 A vector in a coordinate system (a) in two dimensions (b) in three dimensions

We denote the vectors of magnitude 1 in the directions of the x -, y - and z -axes by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, respectively, as shown in Figure 41. These vectors are called the **Cartesian unit vectors**. In general, a **unit vector** is a vector with magnitude 1.

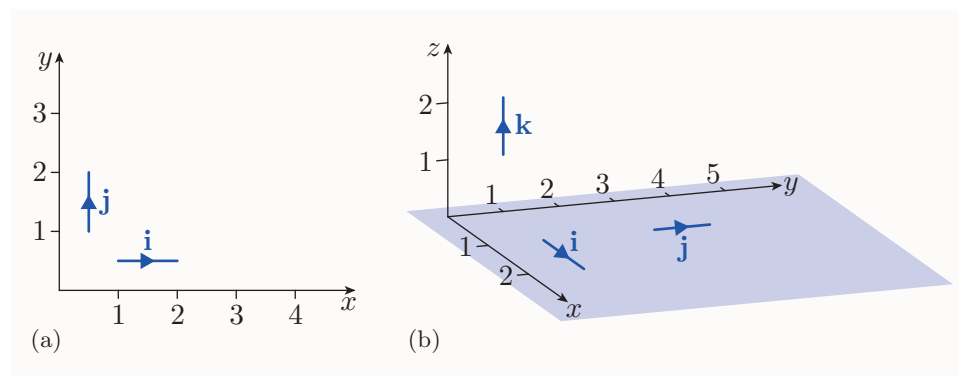


Figure 41 The Cartesian unit vectors (a) in two dimensions (b) in three dimensions

Every vector can be expressed as the sum of scalar multiples of the Cartesian unit vectors. For example, in Figure 42(a), $\mathbf{u} = 3\mathbf{i} + (-2\mathbf{j}) = 3\mathbf{i} - 2\mathbf{j}$, and in Figure 42(b), $\mathbf{v} = \frac{3}{2}\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

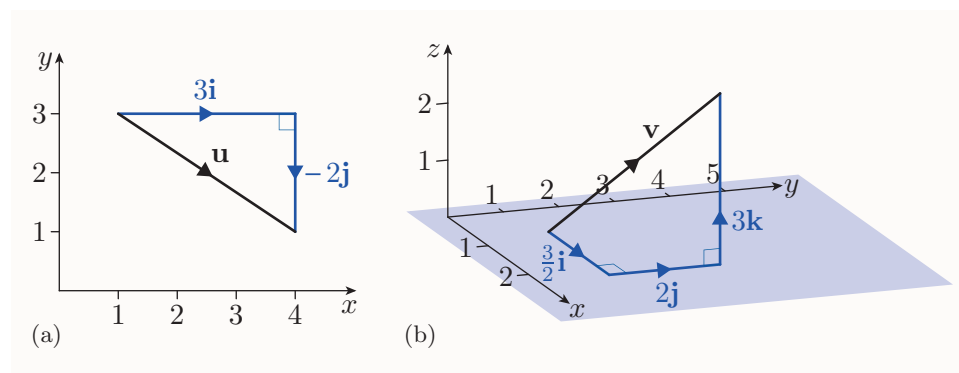


Figure 42 Vectors expressed as sums of scalar multiples of the Cartesian unit vectors

We make the following definitions, illustrated in Figure 43.

Component form of a vector

If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, then the expression $a\mathbf{i} + b\mathbf{j}$ is called the **component form** of $\mathbf{v}$. The scalars a and b are called the **i-component** and **j-component**, respectively, of $\mathbf{v}$.

Similarly, if $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then the expression $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is called the **component form** of $\mathbf{v}$. The scalars a , b and c are called the **i-component**, **j-component** and **k-component**, respectively, of $\mathbf{v}$.

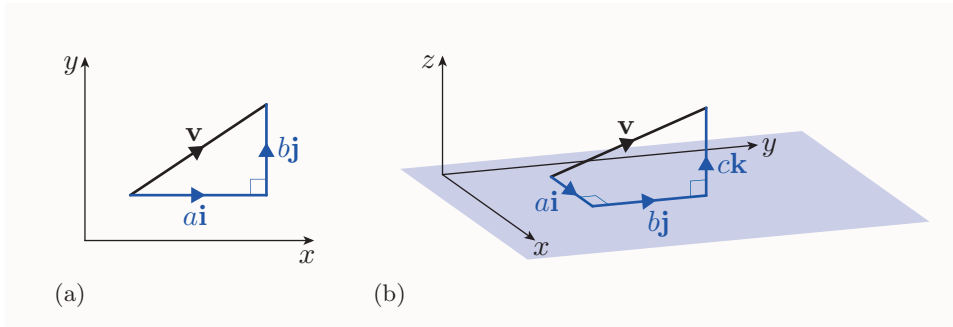


Figure 43 General vectors expressed as sums of scalar multiples of the Cartesian unit vectors

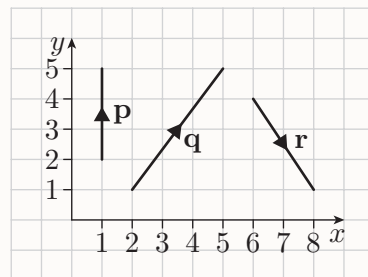
For example, the component form of the vector $\mathbf{u}$ in Figure 42 on the previous page is $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$, and its $\mathbf{i}$ - and $\mathbf{j}$ - components are 3 and -2 , respectively.

The components of a vector represent the ‘amount of the vector’ in the positive direction of each axis.

The $\mathbf{i}$ - and $\mathbf{j}$ -components of a two-dimensional vector are alternatively called the x -component and y -component, respectively. Similarly, the $\mathbf{i}$ -, $\mathbf{j}$ - and $\mathbf{k}$ -components of a three-dimensional vector are alternatively called the x -component, y -component and z -component, respectively.

Activity 34 *Expressing vectors in component form*

Express each of the vectors in the diagram below in component form.



There is a useful alternative notation for expressing a vector in component form, as follows.

Alternative component form of a vector

The vector $a\mathbf{i} + b\mathbf{j}$ can be written as $\begin{pmatrix} a \\ b \end{pmatrix}$.

The vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ can be written as $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

A vector written in this form is called a **column vector**.

For example,

$$3\mathbf{i} - 2\mathbf{j} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

The first number in a column vector is its **i**-component, the second number is its **j**-component, and, in three dimensions, the third number is its **k**-component.

Activity 35 *Writing vectors as column vectors*

Write the vectors in Activity 34 as column vectors.

In general, if you are asked to write a vector in component form, then you can use either of the two types of component form. Both forms are used in this module. Sometimes one form is more convenient than the other.

Notice that, in two dimensions,

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and, in three dimensions,

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The zero vector, like any other vector, can be expressed in component form. In two dimensions,

$$\mathbf{0} = 0\mathbf{i} + 0\mathbf{j} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and, in three dimensions,

$$\mathbf{0} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It is often convenient to denote the components of a vector by the same letter as the vector (but not bold or underlined), with subscripts. For example, we can write

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \text{or} \quad \mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

If a vector represents a displacement whose magnitude is measured in metres, for example, then its components are also measured in metres. Similarly, if a vector represents a velocity whose magnitude is measured in m s^{-1} , for example, then its components are also measured in m s^{-1} .

6.2 Vector algebra using components

In Subsection 5.1 you saw how to add and subtract vectors using the triangle law, and how to multiply a vector by a scalar. Here you will see how to carry out these operations when vectors are expressed in component form.

Addition of vectors

By the properties of vector algebra that you met in Subsection 5.2, if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= a_1\mathbf{i} + a_2\mathbf{j} + b_1\mathbf{i} + b_2\mathbf{j} \\ &= (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}. \end{aligned}$$

This is illustrated in Figure 44.

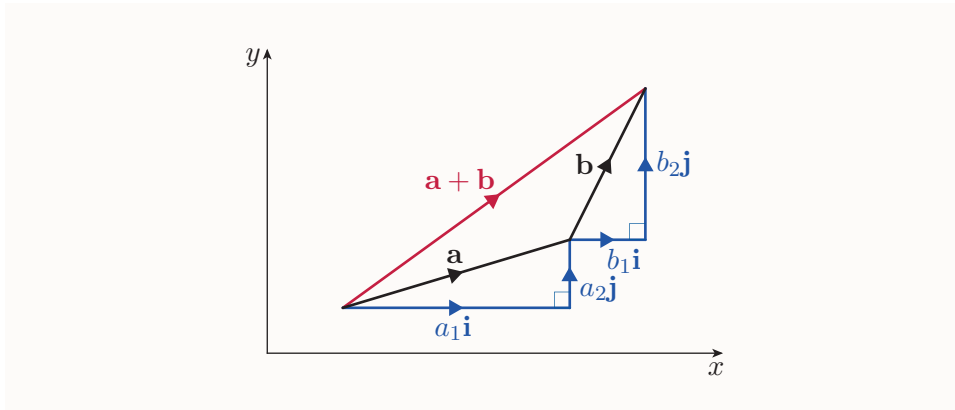


Figure 44 The sum $\mathbf{a} + \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$

So, to add two two-dimensional vectors, you just add their corresponding components.

Addition of two-dimensional vectors in component form

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}.$$

In column notation,

$$\text{if } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \text{ then } \mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}.$$

For example,

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 + 3 \\ 3 + (-1) \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

Vectors in three dimensions are added in a similar way, as follows.

Addition of three-dimensional vectors in component form

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}.$$

In column notation,

$$\text{if } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \text{ then } \mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}.$$

The facts in the boxes above extend to sums of more than two vectors. That is, the $\mathbf{i}$ -component of the sum of several vectors is the sum of the $\mathbf{i}$ -components of the individual vectors, and similarly for the other components.

Example 14 Adding vectors in component form

Let $\mathbf{a} = 4\mathbf{i} + \mathbf{j}$, $\mathbf{b} = -3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} - 2\mathbf{j}$. Find the vector $\mathbf{a} + \mathbf{b} + \mathbf{c}$.

Solution

☁ Add corresponding components. ☁

$$\begin{aligned}\mathbf{a} + \mathbf{b} + \mathbf{c} &= (4\mathbf{i} + \mathbf{j}) + (-3\mathbf{i} + 2\mathbf{j}) + (2\mathbf{i} - 2\mathbf{j}) \\ &= 4\mathbf{i} - 3\mathbf{i} + 2\mathbf{i} + \mathbf{j} + 2\mathbf{j} - 2\mathbf{j} \\ &= 3\mathbf{i} + \mathbf{j}.\end{aligned}$$

Activity 36 Adding vectors in component form

Find the following vector sums.

(a) $(4\mathbf{i} - 2\mathbf{j}) + (-3\mathbf{i} + \mathbf{j})$ (b) $\begin{pmatrix} 5 \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \end{pmatrix}$

(c) $\begin{pmatrix} -7 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}$

Negative of a vector in component form

By the properties of vector algebra, if $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then $-\mathbf{b} = -b_1\mathbf{i} - b_2\mathbf{j}$. This is illustrated in Figure 45.

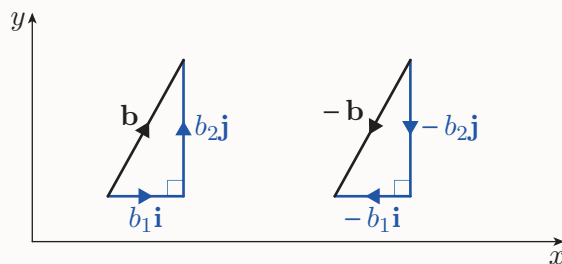


Figure 45 A vector $\mathbf{b}$ and its negative $-\mathbf{b}$

In general we have the following.

Negative of a two-dimensional vector in component form

If $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then $-\mathbf{b} = -b_1\mathbf{i} - b_2\mathbf{j}$.

In column notation,

$$\text{if } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \text{ then } -\mathbf{b} = \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix}.$$

Negative of a three-dimensional vector in component form

If $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then $-\mathbf{b} = -b_1\mathbf{i} - b_2\mathbf{j} - b_3\mathbf{k}$.

In column notation,

$$\text{if } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \text{ then } -\mathbf{b} = \begin{pmatrix} -b_1 \\ -b_2 \\ -b_3 \end{pmatrix}.$$

Subtraction of vectors in component form

By the properties of vector algebra, if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then

$$\begin{aligned} \mathbf{a} - \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j}) - (b_1\mathbf{i} + b_2\mathbf{j}) \\ &= (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j}. \end{aligned}$$

So, to subtract one two-dimensional vector in component form from another, you subtract each component of the first vector from the corresponding component of the other vector.

Subtraction of two-dimensional vectors in component form

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then

$$\mathbf{a} - \mathbf{b} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j}.$$

In column notation,

$$\text{if } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \text{ then } \mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}.$$

A similar fact holds in three dimensions.

Subtraction of three-dimensional vectors in component form

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a} - \mathbf{b} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j} + (a_3 - b_3)\mathbf{k}.$$

In column notation,

$$\text{if } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \text{ then } \mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix}.$$

Example 15 *Subtracting vectors in component form*

Let $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = -\mathbf{i} + 3\mathbf{j}$. Find $\mathbf{a} - \mathbf{b}$.

Solution

 Subtract the corresponding components. 

$$\begin{aligned} \mathbf{a} - \mathbf{b} &= (5\mathbf{i} - 2\mathbf{j}) - (-\mathbf{i} + 3\mathbf{j}) \\ &= 5\mathbf{i} + \mathbf{i} - 2\mathbf{j} - 3\mathbf{j} \\ &= 6\mathbf{i} - 5\mathbf{j}. \end{aligned}$$

Activity 37 *Subtracting vectors in component form*

Find the following vectors.

(a) $(2\mathbf{i} + \mathbf{j}) - (3\mathbf{i} + 2\mathbf{j})$ (b) $(3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) - (-2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$

(c) $\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Multiplication of vectors by scalars

By the properties of vector algebra, if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and m is a scalar, then

$$\begin{aligned} m\mathbf{a} &= m(a_1\mathbf{i} + a_2\mathbf{j}) \\ &= ma_1\mathbf{i} + ma_2\mathbf{j}. \end{aligned}$$

This is illustrated in Figure 46.

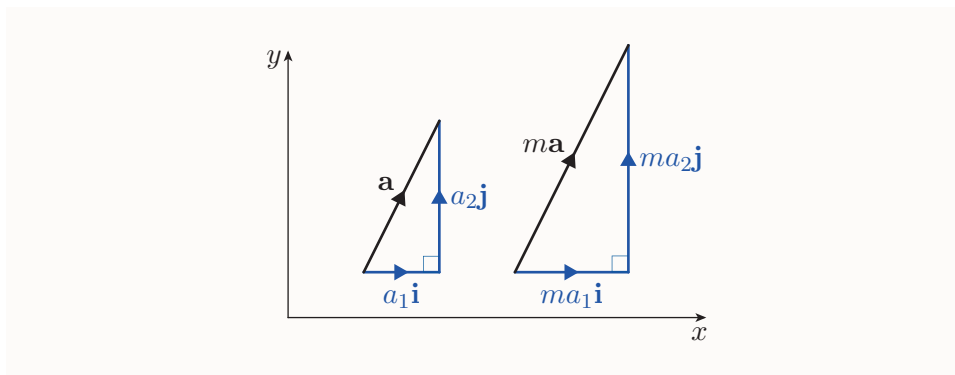


Figure 46 Scaling the vector $\mathbf{a}$ by m

So, to multiply a two-dimensional vector by a scalar m , you simply multiply each component by m .

Scalar multiplication of a two-dimensional vector in component form

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and m is a scalar, then

$$m\mathbf{a} = ma_1\mathbf{i} + ma_2\mathbf{j}.$$

In column notation,

$$\text{if } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \text{ then } m\mathbf{a} = \begin{pmatrix} ma_1 \\ ma_2 \end{pmatrix}.$$

For example, $3(4\mathbf{i} - 5\mathbf{j}) = 12\mathbf{i} - 15\mathbf{j}$.

A similar result applies in three dimensions.

Scalar multiplication of a three-dimensional vector in component form

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and m is a scalar, then

$$m\mathbf{a} = ma_1\mathbf{i} + ma_2\mathbf{j} + ma_3\mathbf{k}.$$

In column notation,

$$\text{if } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \text{ then } m\mathbf{a} = \begin{pmatrix} ma_1 \\ ma_2 \\ ma_3 \end{pmatrix}.$$

Activity 38 *Multiplying a vector in component form by a scalar*

Let $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$.

Find each of the following scalar multiples.

- (a) $4\mathbf{a}$ (b) $-2\mathbf{a}$ (c) $\frac{1}{2}\mathbf{a}$ (d) $3\mathbf{b}$ (e) $-4\mathbf{b}$ (f) $\frac{1}{3}\mathbf{b}$

Combining vector operations

In the next example, the vector operations of addition, subtraction and scalar multiplication are combined.

Example 16 *Simplifying a combination of vectors in component form*

Let $\mathbf{t} = 5\mathbf{i} + 3\mathbf{j}$, $\mathbf{u} = -2\mathbf{i} + 7\mathbf{j}$ and $\mathbf{v} = 4\mathbf{i} - 4\mathbf{j}$. Find $3\mathbf{t} - \mathbf{u} - 5\mathbf{v}$ in component form.

Solution

 Substitute in the expressions for $\mathbf{t}$, $\mathbf{u}$ and $\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$, and simplify. 

$$\begin{aligned} 3\mathbf{t} - \mathbf{u} - 5\mathbf{v} &= 3(5\mathbf{i} + 3\mathbf{j}) - (-2\mathbf{i} + 7\mathbf{j}) - 5(4\mathbf{i} - 4\mathbf{j}) \\ &= 15\mathbf{i} + 9\mathbf{j} + 2\mathbf{i} - 7\mathbf{j} - 20\mathbf{i} + 20\mathbf{j} \\ &= -3\mathbf{i} + 22\mathbf{j}. \end{aligned}$$

Activity 39 Simplifying combinations of vectors in component form

Find each of the following vectors in component form.

(a) $-2\mathbf{a} + 3\mathbf{b} + 4\mathbf{c}$, where $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = \mathbf{i} - 4\mathbf{j}$ and $\mathbf{c} = -5\mathbf{i} + 7\mathbf{j}$

(b) $2 \begin{pmatrix} 6 \\ -3 \end{pmatrix} - 7 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 5 \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

(c) $a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, where a_1 and a_2 are any scalars

(d) $2 \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

6.3 Position vectors

If P is any point, either in the plane or in three-dimensional space, then the **position vector** of P is the displacement vector $\overrightarrow{OP}$, where O is the origin. This is illustrated in Figure 47, in the case of two dimensions.

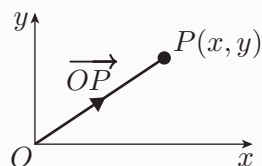


Figure 47 The position vector $\overrightarrow{OP}$ in two dimensions

The components of the position vector of a point are the same as the coordinates of the point. That is, the position vector of the point $P(x, y)$ in two dimensions is

$$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} = \begin{pmatrix} x \\ y \end{pmatrix},$$

and similarly the position vector of the point $Q(x, y, z)$ in three dimensions is

$$\overrightarrow{OQ} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

If a point is denoted by a capital letter, as is usual, then it's often convenient to denote its position vector by the corresponding lower-case, bold (or underlined) letter. For example, we often denote the position vector of the point P by $\mathbf{p}$, the position vector of the point A by $\mathbf{a}$, and so on.

There is a simple equation that expresses any displacement vector $\overrightarrow{AB}$ in terms of the position vectors of the points A and B . Consider Figure 48, which shows two points A and B , the displacement vector $\overrightarrow{AB}$, and the position vectors of A and B .

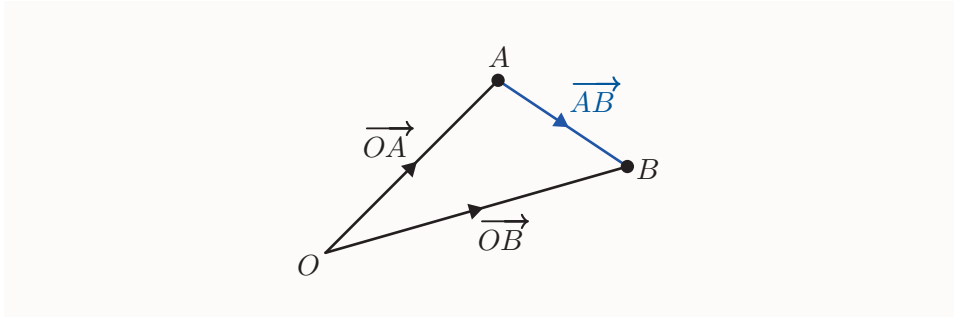


Figure 48 A vector $\overrightarrow{AB}$, and the position vectors of A and B

Since

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB},$$

it follows that

$$\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB};$$

that is,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}.$$

This equation, which applies in either two or three dimensions, can be stated as below.

If the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, then

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}.$$

For example, Figure 49 shows two points A and B , with position vectors $2\mathbf{i} + \mathbf{j}$ and $3\mathbf{i} + 5\mathbf{j}$, respectively. The vector $\overrightarrow{AB}$ is $(3\mathbf{i} + 5\mathbf{j}) - (2\mathbf{i} + \mathbf{j}) = \mathbf{i} + 4\mathbf{j}$.

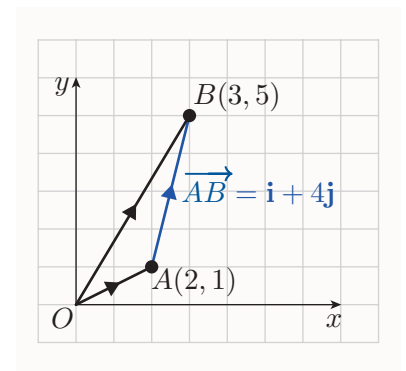


Figure 49 A particular vector $\overrightarrow{AB}$, and the position vectors of A and B

Activity 40 Using position vectors

Consider the points $A(5, 3)$ and $B(-2, 4)$. Find the vector $\overrightarrow{AB}$ in component form.

Vectors, and position vectors in particular, often provide a convenient means of solving problems and proving facts in coordinate geometry. The next example gives an alternative proof of the formula that you met earlier for the coordinates of the midpoint of a line segment in terms of the coordinates of the endpoints.

**Example 17** Working with position vectors

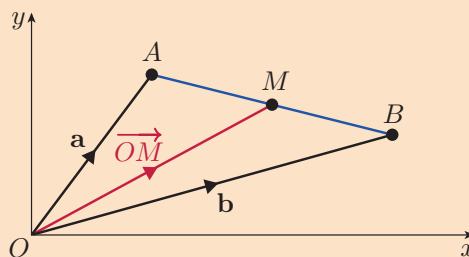
Consider the points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let M be the midpoint of the line segment AB , and let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{m}$ be the position vectors of A , B and M , respectively.

- Express $\mathbf{m}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
- Hence find the coordinates of M , in terms of the coordinates of A and B .

Solution

(a)

First draw a diagram.



We have

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM};$$

that is,

$$\mathbf{m} = \mathbf{a} + \overrightarrow{AM}.$$

Also, $\overrightarrow{AM}$ has the same direction as $\overrightarrow{AB}$ but half its magnitude. So

$$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\mathbf{b} - \mathbf{a}).$$

Hence

$$\begin{aligned}\mathbf{m} &= \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) \\ &= \mathbf{a} + \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} \\ &= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} \\ &= \frac{1}{2}(\mathbf{a} + \mathbf{b}).\end{aligned}$$

(b)

 Use the components of $\mathbf{a}$ and $\mathbf{b}$ to find the components of $\mathbf{m}$ and hence the coordinates of M . 

Since $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j}$, we have

$$\begin{aligned}\mathbf{m} &= \frac{1}{2}(x_1\mathbf{i} + y_1\mathbf{j} + x_2\mathbf{i} + y_2\mathbf{j}) \\ &= \left(\frac{x_1 + x_2}{2}\right)\mathbf{i} + \left(\frac{y_1 + y_2}{2}\right)\mathbf{j}.\end{aligned}$$

Hence the coordinates of M are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Example 17 gives the following useful fact.

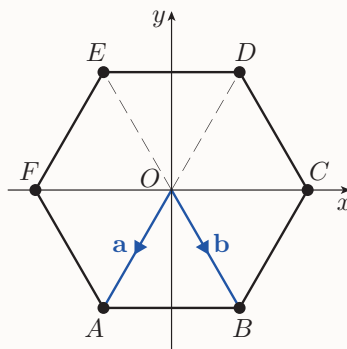
Midpoint formula in terms of position vectors

If the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, then the midpoint of the line segment AB has position vector $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.

Here is a similar example for you to try.

Activity 41 Working with position vectors

Consider the regular hexagon $ABCDEF$, with centre the origin, shown below. Here $\vec{OA} = \mathbf{a}$ and $\vec{OB} = \mathbf{b}$.



Express the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$.

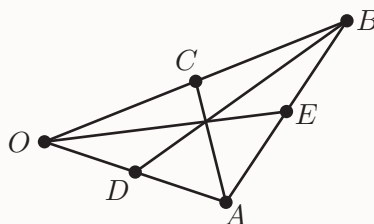
- (a) $\vec{AB}$ (b) $\vec{OC}$ (c) $\vec{BC}$ (d) $\vec{AD}$ (e) $\vec{AE}$

(Remember that each of the six triangles OAB , OBC , OCD , ODE , OEF and OFA is equilateral.)

In the next activity you are asked to use position vectors to prove the following basic property of every triangle: the three lines that each join a vertex to the midpoint of the opposite side meet at a common point.

Activity 42 Proving a geometric property of triangles

In the triangle OAB shown below, the points C , D and E are the midpoints of the sides of the triangle. Let $\vec{OA} = \mathbf{a}$ and $\vec{OB} = \mathbf{b}$.



- (a) Show that

$$\vec{AC} = \frac{1}{2}\mathbf{b} - \mathbf{a} \quad \text{and} \quad \vec{BD} = \frac{1}{2}\mathbf{a} - \mathbf{b}.$$

- (b) Find the position vectors of each of the following points, expressed in terms of the position vectors $\mathbf{a}$ and $\mathbf{b}$.
- (i) The point two-thirds of the way along OE from O .
 - (ii) The point two-thirds of the way along AC from A .
 - (iii) The point two-thirds of the way along BD from B .
- (c) Deduce that the lines OE , AC and BD are concurrent; that is, they meet at a common point.

6.4 Converting from component form to magnitude and direction, and vice versa

In this subsection you will learn how to find the magnitude and direction of a two-dimensional vector from its components, and how to find the components of a two-dimensional vector from its magnitude and direction. These skills will enable you to use components to solve problems specified in terms of magnitudes and directions. You will see some examples of this in Subsection 6.5.

You'll also learn here how to calculate the magnitude of a three-dimensional vector from its components. Techniques involving the directions of three-dimensional vectors are more complicated, and are not covered in this module.

Finding the magnitude of a vector from its components

Consider any two-dimensional vector $\mathbf{v}$, not parallel to an axis. If you draw a right-angled triangle whose hypotenuse is $\mathbf{v}$, and whose two shorter sides are parallel to the axes, then the lengths of these two shorter sides are the *magnitudes* of the components of $\mathbf{v}$. This is illustrated in Figure 50, for a vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ whose components a and b are positive.

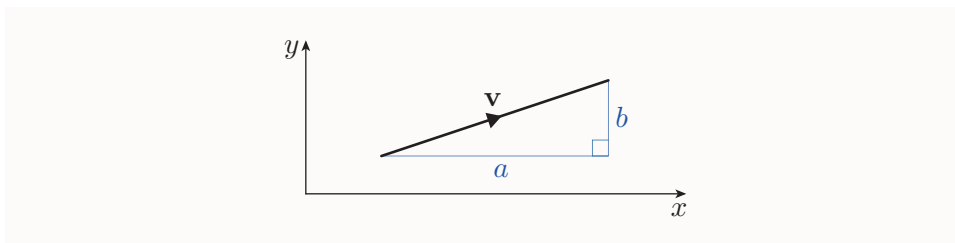


Figure 50 A two-dimensional vector with positive components

By Pythagoras' theorem, the magnitude of the vector $\mathbf{v}$ in Figure 50 is given by

$$|\mathbf{v}| = \sqrt{a^2 + b^2}.$$

You can see that this formula will hold for any vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, no matter whether the components a and b are positive, negative or zero, for reasons similar to those that you saw for the distance formula in Subsection 1.1. So we have the following fact.

The magnitude of a two-dimensional vector in terms of its components

The magnitude of the vector $a\mathbf{i} + b\mathbf{j}$ is $\sqrt{a^2 + b^2}$.

There is a similar formula for three-dimensional vectors.

The magnitude of a three-dimensional vector in terms of its components

The magnitude of the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is $\sqrt{a^2 + b^2 + c^2}$.

Activity 43 *Finding the magnitudes of vectors from their components*

Find the magnitudes of the following vectors. Give exact answers.

- (a) $-3\mathbf{i} + \mathbf{j}$ (b) $2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ (c) $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ (d) $\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$

Finding the direction of a two dimensional vector from its components

As you saw earlier, you can specify the direction of a two-dimensional vector by stating the angle (measured clockwise or anticlockwise) from some chosen reference direction to the direction of the vector.

You saw that one way to do this, which is common in air and maritime navigation, is to use *bearings*. To express the direction of a vector as a bearing, you state the angle measured clockwise from north to the direction of the vector. In this module, bearings are always stated as angles between 0° and 360° .

Another method of expressing the direction of a two-dimensional vector, which is often used when components are involved, is to state the angle measured from the positive x -direction to the direction of the vector. The angle is always measured *anticlockwise*, and this is usually assumed rather than being stated explicitly. This way of measuring angles is the standard way that you met in Unit 4. For example, Figure 51 shows two vectors making angles of 35° and 200° , respectively, with the positive x -direction.

With this method, you can state the angles in any convenient way – they don't need to be between 0° and 360° , and they could be measured in radians, rather than degrees. Sometimes it is helpful to use negative angles – for example, you can say that the vector **b** in Figure 51 makes an angle of -160° with the positive direction of the x -axis.

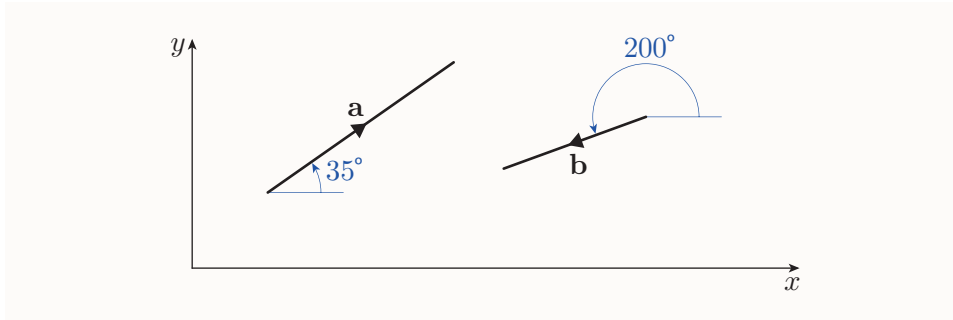


Figure 51 A vector **a** that makes an angle of 35° with the positive x -direction, and a vector **b** that makes an angle of 200° with the positive x -direction

Suppose that you have a two-dimensional vector in component form, and you want to find the angle that it makes with some reference direction, such as north or the positive x -direction. The first thing to do is to sketch the vector. If it is parallel to an axis, then it should be straightforward to find the angle that you want. Otherwise, you can sketch a right-angled triangle whose hypotenuse is the vector, and whose shorter sides are parallel to the axes. The lengths of the shorter sides are the magnitudes of the components of the vector.

You can use basic trigonometry to find an acute angle in the triangle, and you can then use this acute angle to find the angle that you want. This method is demonstrated in the following example.


Example 18 *Finding the magnitude and direction of a vector from its components*

Find the magnitude of the vector $4\mathbf{i} - 3\mathbf{j}$, and the angle that it makes with the positive x -direction. Give the angle to the nearest degree.

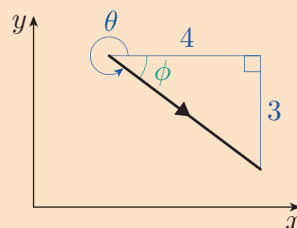
Solution

Use the standard formula to find the magnitude.

The magnitude of the vector is

$$\sqrt{4^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$$

To find the required angle, first draw a diagram.



Find the acute angle ϕ , and hence find the required angle.

From the diagram,

$$\tan \phi = \frac{3}{4},$$

so

$$\phi = \tan^{-1} \frac{3}{4} = 36.86 \dots^\circ.$$

Hence the angle, labelled θ , that the vector makes with the positive x -direction is

$$\begin{aligned} 360^\circ - 36.86 \dots^\circ &= 323.13 \dots^\circ \\ &= 323^\circ \text{ (to the nearest degree).} \end{aligned}$$

Activity 44 Finding magnitudes and directions of vectors from their components

Find the magnitudes and directions of the following vectors. Give the magnitudes as exact values, and give the directions as the angles that the vectors make with the positive x -direction, to the nearest degree.

(a) $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j}$ (b) $\mathbf{b} = \begin{pmatrix} 0 \\ -3.5 \end{pmatrix}$

To find the direction of a vector as a *bearing*, you can use methods similar to those in Example 18. The only difference is that in the last step you need to find the angle measured clockwise from north to the direction of the vector. You can try this in the next activity.

Activity 45 Finding magnitudes, and directions as bearings, from components

Find the magnitude and bearing of each of the following velocity vectors. Assume that $\mathbf{i}$ and $\mathbf{j}$ are taken to point east and north, respectively, and that the units are ms^{-1} . Give the magnitudes of the vectors in ms^{-1} to one decimal place, and their bearings to the nearest degree.

(a) $\mathbf{u} = 4\mathbf{i} - 2\mathbf{j}$ (b) $\mathbf{v} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$

Finding the components of a two-dimensional vector from its magnitude and direction

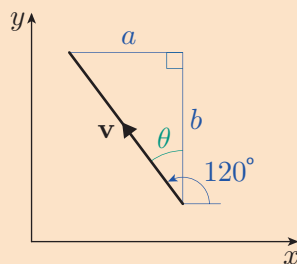
To find the components of a two-dimensional vector from its magnitude and direction, again the key is to first sketch the vector. If it is parallel to an axis, then it is straightforward to find its components. Otherwise, as before, you can sketch a right-angled triangle whose hypotenuse is the vector, and whose shorter sides are parallel to the axes. You can use basic trigonometry in this triangle to find the *magnitudes* of the components, and you can find their *signs* by using the direction of the vector. This method is demonstrated in the following example.


Example 19 Finding the components of a vector from its magnitude and direction

Express in component form the vector $\mathbf{v}$ with magnitude 4 that makes an angle of 120° with the positive x -direction.

Solution

First sketch the vector and a right-angled triangle whose shorter sides represent the magnitudes of the components.



Find the size of an acute angle in the triangle.

The angle marked θ is given by $\theta = 120^\circ - 90^\circ = 30^\circ$.

Calculate the side lengths a and b , which are the magnitudes of the components.

From the triangle,

$$\sin \theta = \frac{a}{|\mathbf{v}|}.$$

So, since $|\mathbf{v}| = 4$,

$$a = |\mathbf{v}| \sin \theta = 4 \sin 30^\circ = 4 \times \frac{1}{2} = 2.$$

Similarly,

$$\cos \theta = \frac{b}{|\mathbf{v}|}.$$

So

$$b = |\mathbf{v}| \cos \theta = 4 \cos 30^\circ = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}.$$

Find the signs of the components. You can see from the diagram that the $\mathbf{i}$ -component is negative, and the $\mathbf{j}$ -component is positive. Hence write the vector in component form.

Hence

$$\mathbf{v} = -2\mathbf{i} + 2\sqrt{3}\mathbf{j}.$$

 You can check that the magnitude of this vector is as given in the question. 

Here are some vectors for you to express in component form.

Activity 46 *Calculating the components of vectors from their magnitudes and directions*

Express the following vectors in component form, giving each component to one decimal place.

- (a) The vector $\mathbf{p}$ with magnitude 1 that makes an angle of 45° with the positive x -direction.
- (b) The vector $\mathbf{q}$ with magnitude 2.5 that makes an angle of 340° with the positive x -direction.

In the next activity the directions of the vectors are given as bearings.

Activity 47 *Calculating the components of more vectors from their magnitudes and directions*

Express the following vectors in component form. Assume that $\mathbf{i}$ and $\mathbf{j}$ are taken to point east and north, respectively. Give each component to one decimal place.

- (a) The displacement vector $\mathbf{p}$, with magnitude 3.5 m, and bearing 340° .
- (b) The velocity vector $\mathbf{u}$, with magnitude 5.2 m s^{-1} and bearing 240° .

There is a useful standard formula for calculating the components of a vector from its magnitude and direction, when its direction is given as the angle that it makes with the positive x -direction. You can use this formula instead of the method in Example 19, when it is convenient to do so.

To obtain this formula, consider any non-zero vector $\mathbf{v}$, and let the angle that it makes with the positive x -direction be θ .

Let $\mathbf{u}$ be the vector in the same direction as $\mathbf{v}$, but with magnitude 1. If you draw $\mathbf{u}$ with its tail at the origin, as shown in Figure 52(a), then its tip lies on the unit circle, and hence, by what you saw in Subsection 2.2 of Unit 4, its tip has coordinates $(\cos \theta, \sin \theta)$. This is true whatever the size of the angle θ .

Hence $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$. Multiplying $\mathbf{u}$ by the magnitude of the vector $\mathbf{v}$ gives the vector $\mathbf{v}$, so

$$\mathbf{v} = |\mathbf{v}|(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) = |\mathbf{v}| \cos \theta \mathbf{i} + |\mathbf{v}| \sin \theta \mathbf{j},$$

as illustrated in Figure 52(b). This is a formula for the components of $\mathbf{v}$.

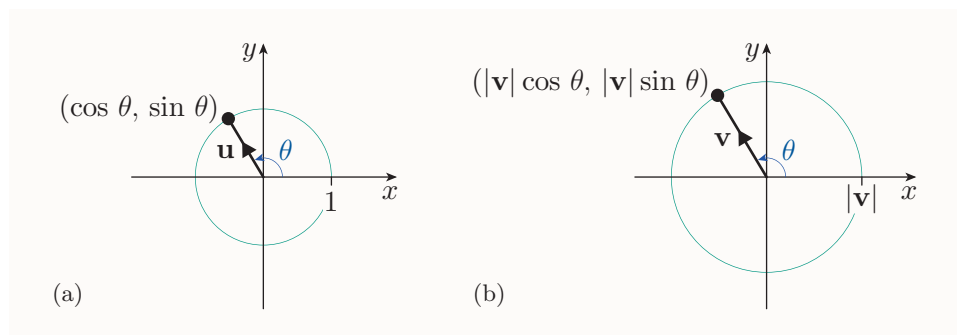


Figure 52 (a) A vector $\mathbf{u}$ of magnitude 1 with its tail at the origin (b) a vector $\mathbf{v}$ of any magnitude, in the same direction as $\mathbf{u}$, with its tail at the origin

So we have the following general fact.

Component form of a two-dimensional vector in terms of its magnitude and its angle with the positive x -direction

If the two-dimensional vector $\mathbf{v}$ makes the angle θ with the positive x -direction, then

$$\mathbf{v} = |\mathbf{v}| \cos \theta \mathbf{i} + |\mathbf{v}| \sin \theta \mathbf{j}.$$

This fact is illustrated in Figure 53.

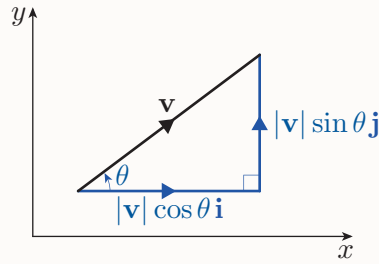


Figure 53 A vector $\mathbf{v}$ and its components

You can practise using this formula in the next activity. Remember that where the direction of a vector is given as a bearing, you need to start by finding the angle that it makes with the positive x -direction.

Activity 48 *Using the formula for the component form of a vector*

Use the formula above to find the component forms of the following vectors. In part (b), assume that $\mathbf{i}$ and $\mathbf{j}$ are taken to point east and north, respectively. Give each component to two significant figures.

- (a) The vector $\mathbf{a}$ with magnitude 78 that makes an angle of 216° with the positive x -direction.
- (b) The velocity vector $\mathbf{w}$ with magnitude 4.4 m s^{-1} and bearing 119° .

6.5 Using vectors in component form

When you're using vectors to help you solve a practical problem, you often have to find the sum of two or more vectors. If you know the magnitudes and directions of the vectors, then you can do this by using the triangle rule for vector addition and applying trigonometry, as you saw in Subsection 5.3.

However, as you saw, the calculations involved can be complicated. It is often easier to write the vectors in component form, add them in this form, and then find the magnitude and direction of the resultant vector.

This method is demonstrated in the next example.



Example 20 Combining displacements using components

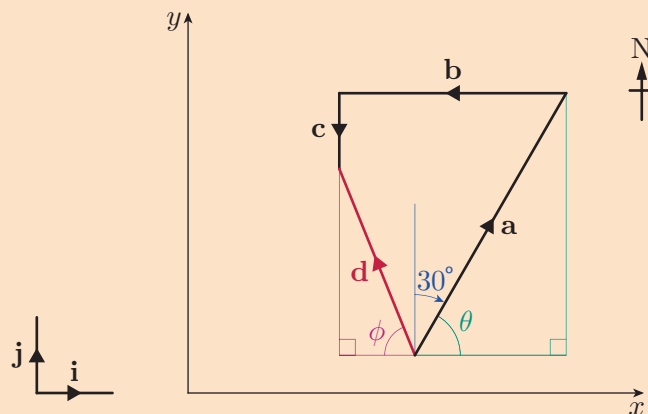
A metal detectorist walks 200 m on a bearing of 30° , turns and walks 150 m due west, then walks 50 m due south.

Find the magnitude and bearing of his resultant displacement from his initial position. Give the magnitude to the nearest metre, and the bearing to the nearest degree.

Solution

Let the three parts of the metal detectorist's walk be the vectors **a**, **b** and **c**, and let the resultant displacement be the vector **d**.

Choose a convenient coordinate system. Since the problem involves bearings, choose the *y*-axis to point north and the *x*-axis to point east. Draw a diagram showing the vectors. Mark any angles that you know, and mark or state any magnitudes that you know.



We know that $|\mathbf{a}| = 200$ m, $|\mathbf{b}| = 150$ m and $|\mathbf{c}| = 50$ m.

Express the vectors representing the parts of the walk in component form. To find the components of **a**, first find the angle marked θ .

In the diagram, $\theta = 90^\circ - 30^\circ = 60^\circ$. Hence

$$\begin{aligned}\mathbf{a} &= |\mathbf{a}| \cos \theta \mathbf{i} + |\mathbf{a}| \sin \theta \mathbf{j} \\ &= 200 \cos 60^\circ \mathbf{i} + 200 \sin 60^\circ \mathbf{j} \\ &= 100\mathbf{i} + 100\sqrt{3}\mathbf{j}.\end{aligned}$$

Also, $\mathbf{b} = -150\mathbf{i}$ and $\mathbf{c} = -50\mathbf{j}$.

 Find the resultant displacement $\mathbf{d}$, and hence calculate its magnitude and direction. 

Hence the resultant displacement $\mathbf{d}$ is

$$\begin{aligned}\mathbf{d} &= \mathbf{a} + \mathbf{b} + \mathbf{c} \\ &= 100\mathbf{i} + 100\sqrt{3}\mathbf{j} - 150\mathbf{i} - 50\mathbf{j} \\ &= -50\mathbf{i} + (100\sqrt{3} - 50)\mathbf{j}.\end{aligned}$$

The magnitude of $\mathbf{d}$ is given by

$$|\mathbf{d}| = \sqrt{(-50)^2 + (100\sqrt{3} - 50)^2} = 132.964 \dots \text{ m}.$$

The angle ϕ shown on the diagram can be found by using the components of $\mathbf{d}$, as follows:

$$\tan \phi = \frac{100\sqrt{3} - 50}{50},$$

so

$$\phi = \tan^{-1} \left(\frac{100\sqrt{3} - 50}{50} \right) = 67.91 \dots^\circ.$$

So the bearing of $\mathbf{d}$ is $270^\circ + 67.91 \dots^\circ = 337.91 \dots^\circ$.

 Write down a conclusion, including units. 

The resultant displacement has magnitude 133 m (to the nearest metre) and bearing 338° (to the nearest degree).

Here are two problems for you to try. You might recognise the first of them: it was solved in Example 13 on page 152 by using the triangle law for vector addition. You should find that using components is less complicated, but gives the same answers!

Activity 49 *Combining displacements using components*

The displacement from Exeter to Belfast is 460 km with a bearing of 340° , and the displacement from Belfast to Glasgow is 173 km with a bearing of 36° . By using components, find the magnitude (to the nearest kilometre) and direction (as a bearing, to the nearest degree) of the displacement from Exeter to Glasgow.

Activity 50 Combining velocities using components

An aircraft flies on a course with a bearing of 80° , and with a horizontal ground speed of 600 mph. (This is its horizontal speed relative to the ground – you can think of it as the speed of a point directly underneath the plane on horizontal ground.)

- Express the horizontal ground velocity of the aircraft in component form. Take $\mathbf{i}$ and $\mathbf{j}$ to point east and north, respectively, and state each component to the nearest mph.
- In addition to this horizontal ground velocity, the aircraft is also climbing at a rate of 30 mph. Find the resultant velocity of the aircraft in component form and the magnitude of the resultant velocity (to the nearest mph).
- The resultant velocity of the aircraft found in part (b) is actually a combination of its velocity relative to the air in which it is flying, which is called its *air velocity*, and the velocity of the air itself, which is called the *wind velocity*. Suppose that the wind velocity is horizontal, with a speed of 100 mph *from* a bearing of 290° . Find the air velocity of the aircraft, in component form, and the magnitude of the air velocity (to the nearest mph).

7 Scalar product of two vectors

In this section you'll learn a way to multiply two vectors, called the *scalar product*, also known as the *dot product*, of the two vectors, and you'll see that this quantity provides a convenient method for finding the angle between any two vectors.

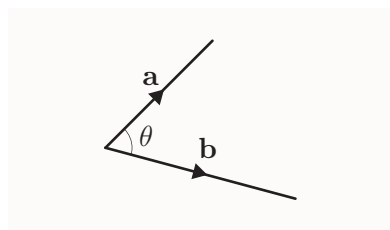


Figure 54 The angle between two vectors $\mathbf{a}$ and $\mathbf{b}$

7.1 Calculating scalar products

First, let's clarify what's meant by the angle between two vectors. The **angle between** two vectors $\mathbf{a}$ and $\mathbf{b}$ is the angle θ in the range $0 \leq \theta \leq 180^\circ$ between their directions when the vectors are placed tail-to-tail, as illustrated in Figure 54. This definition applies to both two-dimensional and three-dimensional vectors.

Here's the definition of the scalar product.

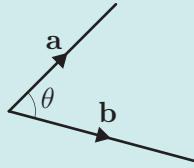
Scalar product of two vectors

The **scalar product** of the non-zero vectors **a** and **b** is

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta,$$

where θ is the angle between **a** and **b**.

If **a** or **b** is the zero vector, then $\mathbf{a} \cdot \mathbf{b} = 0$.



The notation $\mathbf{a} \cdot \mathbf{b}$ is read as ‘a dot b’. The scalar product $\mathbf{a} \cdot \mathbf{b}$ of two vectors **a** and **b** is a *scalar*, since none of the quantities $|\mathbf{a}|$, $|\mathbf{b}|$ and $\cos \theta$ has a direction. This is why it is called the *scalar* product. The alternative name of *dot product* arises from the notation used. When you are writing a scalar product, it is important to make sure that the dot between the vectors is clear.

The definition of scalar product applies to both two-dimensional and three-dimensional vectors. It has a geometric interpretation, as follows. Figure 55 shows two non-zero vectors **a** and **b** placed tail-to-tail, and the right-angled triangle formed by drawing a line from the tip of **a**, perpendicular to **b**. The vector **p** in the diagram is called the **projection** of **a** on **b**. You can see that if θ is an acute angle, then this projection has magnitude $|\mathbf{a}| \cos \theta$, and hence

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{b}| \times (\text{magnitude of projection of } \mathbf{a} \text{ on } \mathbf{b}).$$

Similarly, by considering the projection of **b** on **a**, you can see that

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times (\text{magnitude of projection of } \mathbf{b} \text{ on } \mathbf{a}).$$

If θ is an obtuse angle, then the projection of **a** on **b** points in the opposite direction to **b** (and similarly the projection of **b** on **a** points in the opposite direction to **a**), and $\mathbf{a} \cdot \mathbf{b}$ is the negative of the quantity on the right-hand side of each of the two equations above.

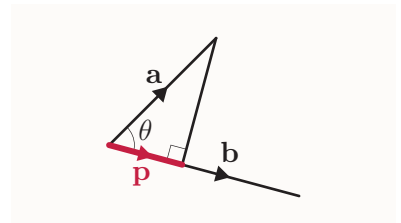
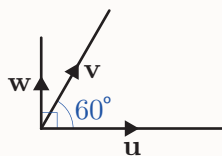


Figure 55 The projection **p** of **a** on **b**

Activity 51 Finding scalar products

Suppose that **u**, **v** and **w** are two-dimensional vectors with magnitudes 4, 3 and 2, respectively, and directions as shown below.



Use the definition of scalar product in the box above to find $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{u}$.

Activity 51 illustrates two important properties of the scalar product.

First, if two non-zero vectors are perpendicular, then their scalar product is zero. This is because if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos 90^\circ = |\mathbf{a}| |\mathbf{b}| \times 0 = 0.$$

It's also true that if the scalar product of two non-zero vectors is zero, then the vectors are perpendicular. This is because if $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors, then the only way that $\mathbf{a} \cdot \mathbf{b}$ can be equal to zero is if $\cos \theta = 0$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. This implies that $\theta = 90^\circ$.

The second property illustrated by Activity 51 is that the scalar product of a vector with itself is equal to the square of the magnitude of the vector. This is obvious for the zero vector, and if $\mathbf{a}$ is any non-zero vector then the angle between $\mathbf{a}$ and itself is 0° , so

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}| |\mathbf{a}| \cos 0^\circ = |\mathbf{a}| |\mathbf{a}| \times 1 = |\mathbf{a}|^2.$$

These, and other, properties of the scalar product are summarised below. They can all be proved using the definition of the scalar product in the box above.

Properties of the scalar product

The following properties hold for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and every scalar m .

1. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are non-zero. If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, then $\mathbf{a} \cdot \mathbf{b} = 0$, and vice versa.
2. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
3. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
4. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
5. $(m\mathbf{a}) \cdot \mathbf{b} = m(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (m\mathbf{b})$

Property 3 states that the operation of taking a scalar product is commutative. Property 4 states that the operation of taking a scalar product is distributive over vector addition. Property 5 states that multiplication by a scalar is distributive over the operation of taking a scalar product.

An alternative version of property 4 in which the vector $\mathbf{a}$ appears *after* the brackets also holds; that is,

$$(\mathbf{b} + \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{a}.$$

You can prove this property by combining property 4 with property 3.

The following example shows how you can use these properties to simplify expressions containing scalar products.

Example 21 *Simplifying an expression containing a scalar product*

Expand and simplify the expression $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors.

Solution

 Expand the brackets by using property 4 in the box above (and the alternative version). 

$$\begin{aligned}(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) &= \mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) + \mathbf{b} \cdot (\mathbf{a} + \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b}\end{aligned}$$

 Simplify by using the fact that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (by property 3). 

$$= \mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}$$

Activity 52 *Simplifying expressions containing a scalar product*

Expand and simplify the expression $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors.

Calculating scalar products from components

You can calculate the scalar product of two vectors directly from the components of the vectors.

Consider the two-dimensional vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$. Their scalar product is

$$\mathbf{a} \cdot \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j}) \cdot (b_1\mathbf{i} + b_2\mathbf{j}).$$

Expanding the brackets and simplifying gives

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (a_1\mathbf{i}) \cdot (b_1\mathbf{i} + b_2\mathbf{j}) + (a_2\mathbf{j}) \cdot (b_1\mathbf{i} + b_2\mathbf{j}) \\ &= (a_1\mathbf{i}) \cdot (b_1\mathbf{i}) + (a_1\mathbf{i}) \cdot (b_2\mathbf{j}) + (a_2\mathbf{j}) \cdot (b_1\mathbf{i}) + (a_2\mathbf{j}) \cdot (b_2\mathbf{j}) \\ &= a_1b_1(\mathbf{i} \cdot \mathbf{i}) + a_1b_2(\mathbf{i} \cdot \mathbf{j}) + a_2b_1(\mathbf{j} \cdot \mathbf{i}) + a_2b_2(\mathbf{j} \cdot \mathbf{j}).\end{aligned}$$

But the vectors $\mathbf{i}$ and $\mathbf{j}$ both have magnitude 1 and are perpendicular to each other, so

$$\mathbf{i} \cdot \mathbf{i} = |\mathbf{i}|^2 = 1, \quad \mathbf{j} \cdot \mathbf{j} = |\mathbf{j}|^2 = 1, \quad \mathbf{i} \cdot \mathbf{j} = 0 \quad \text{and} \quad \mathbf{j} \cdot \mathbf{i} = 0.$$

Substituting into the expression for $\mathbf{a} \cdot \mathbf{b}$ obtained above gives the following result.

Scalar product of two-dimensional vectors in terms of components

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2.$$

In column notation,

$$\text{if } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \text{ then } \mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2.$$

You can obtain a similar result for three-dimensional vectors, as follows.

Scalar product of three-dimensional vectors in terms of components

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

In column notation,

$$\text{if } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \text{ then } \mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

So, to calculate the scalar product of two vectors in component form, you multiply corresponding components and add the results.

Example 22 *Calculating a scalar product using components*

Suppose that $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. Find $\mathbf{a} \cdot \mathbf{b}$.

Solution

 Multiply corresponding components, and add the results. 

$$\mathbf{a} \cdot \mathbf{b} = (2 \times 1) + (1 \times (-2)) + ((-3) \times 2) = -6$$

Here is an example for you to try.

Activity 53 *Calculating scalar products using components*

Suppose that $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{w} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. Find $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{w}$ and $\mathbf{v} \cdot \mathbf{w}$.

7.2 Finding the angle between two vectors

The scalar product of two vectors has an important application in calculating the angle between two vectors.

Rearranging the definition of the scalar product gives the following fact.

Angle between two vectors

The angle θ between any two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|},$$

where $0 \leq \theta \leq 180^\circ$.

The next example demonstrates how to use this fact to find the angle between two vectors in component form.

Example 23 Calculating the angle between two vectors in component form

Find, to the nearest degree, the angle between the vectors $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Solution

 Use the components of the vectors to find $\mathbf{a} \cdot \mathbf{b}$, $|\mathbf{a}|$ and $|\mathbf{b}|$. 

We have

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \\ &= 2 \times 1 + 2 \times 3 + (-1) \times 2 \\ &= 6,\end{aligned}$$

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3,$$

$$|\mathbf{b}| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}.$$

 Apply the result in the box above. 

Hence

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{6}{3\sqrt{14}} = \frac{2}{\sqrt{14}},$$

so

$$\theta = \cos^{-1} \left(\frac{2}{\sqrt{14}} \right) = 57.68 \dots^\circ.$$

That is, the angle between the vectors is 58° , to the nearest degree.

Activity 54 *Calculating the angle between two vectors in component form*

Find, to the nearest degree, the angle between the vectors

$$\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}.$$

The next activity is the final one in this unit. You'll need to apply many of the skills in working with vectors that you have learned.

Activity 55 *Calculating angles between velocities*

A yacht is sailing at 4.8 m s^{-1} on a course of 50° , and a small rowing boat is travelling at 1.1 m s^{-1} on a course of 140° .

- (a) Write down the angle between the paths of the two boats.
- (b) The wind suddenly changes, which results in each boat having an additional velocity imposed on it of 3.5 m s^{-1} with bearing 70° . Calculate the new speed of each boat in m s^{-1} (to one decimal place) and the new angle between their paths (to the nearest degree).

Learning outcomes

After studying this unit, you should be able to:

- calculate the distance between two points in two and three dimensions
- find the midpoint and the perpendicular bisector of the line segment between two points
- recognise the equations of a circle and a sphere, and find the centre and radius of a circle or sphere from its equation
- find the equation of the circle through three points
- calculate the points of intersection of lines, circles and parabolas
- understand what a vector is, and understand the difference between vector and scalar quantities
- find the components of a two-dimensional vector from its magnitude and direction, and vice versa
- find the magnitude of a three-dimensional vector from its components
- add and subtract vectors, and multiply them by a scalar
- use vectors to solve some practical problems involving displacements and velocities
- calculate the scalar product of two vectors, and find the angle between two non-zero vectors.

Solutions to activities

Solution to Activity 1

- (a) The distance between the points $(3, 2)$ and $(7, 5)$ is

$$\sqrt{(7-3)^2 + (5-2)^2} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5.$$

- (b) The distance between the points $(-1, 4)$ and $(3, -2)$ is

$$\begin{aligned}\sqrt{(3-(-1))^2 + (-2-4)^2} &= \sqrt{4^2 + (-6)^2} \\ &= \sqrt{52} = 2\sqrt{13}.\end{aligned}$$

Solution to Activity 2

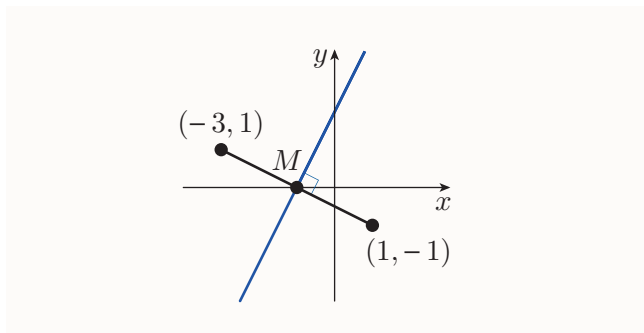
(The solution to this activity is discussed in the text after the activity.)

Solution to Activity 3

The midpoint of the line segment joining $(3, 4)$ and $(5, -3)$ is

$$\left(\frac{3+5}{2}, \frac{4-3}{2}\right) = \left(4, \frac{1}{2}\right).$$

Solution to Activity 4



The midpoint of the line segment joining the points is

$$\left(\frac{-3+1}{2}, \frac{1-1}{2}\right) = (-1, 0).$$

The gradient of the line segment is

$$\frac{1-(-1)}{-3-1} = \frac{2}{-4} = -\frac{1}{2}.$$

So the gradient of the perpendicular bisector is 2.

Hence the equation of the perpendicular bisector is

$$y - 0 = 2(x - (-1)),$$

which simplifies to

$$y = 2x + 2.$$

Solution to Activity 5

- (a) The equation of the circle with centre $(0, 0)$ and radius 3 is $x^2 + y^2 = 9$.

- (b) The equation of the circle with centre $(5, 7)$ and radius $\sqrt{2}$ is

$$(x-5)^2 + (y-7)^2 = 2,$$

since $(\sqrt{2})^2 = 2$.

- (c) The equation of the circle with centre $(-3, -1)$ and radius $\frac{1}{2}$ is

$$(x-(-3))^2 + (y-(-1))^2 = \left(\frac{1}{2}\right)^2;$$

that is,

$$(x+3)^2 + (y+1)^2 = \frac{1}{4}.$$

- (d) The equation of the circle with centre $(3, -\sqrt{3})$ and radius $2/\sqrt{5}$ is

$$(x-3)^2 + (y-(-\sqrt{3}))^2 = \left(\frac{2}{\sqrt{5}}\right)^2;$$

that is,

$$(x-3)^2 + (y+\sqrt{3})^2 = \frac{4}{5}.$$

Solution to Activity 6

- (a) The circle $(x-1)^2 + (y-2)^2 = 25$ has centre $(1, 2)$ and radius $\sqrt{25} = 5$.

- (b) The circle $(x+1)^2 + (y+2)^2 = 49$ has centre $(-1, -2)$ and radius 7.

- (c) The circle $(x-\pi)^2 + (y+\pi)^2 = \pi^2$ has centre $(\pi, -\pi)$ and radius π .

- (d) The given equation is $4x^2 + 4(y-\sqrt{3})^2 = 7$. Dividing by 4, to express it in standard form, gives

$$x^2 + (y-\sqrt{3})^2 = \frac{7}{4},$$

so the centre is $(0, \sqrt{3})$ and the radius is $\sqrt{7/4} = \sqrt{7}/2$.

Solution to Activity 7

- (a) The given equation is

$$x^2 + y^2 - 6x + 8y = 0.$$

Completing the squares gives

$$(x-3)^2 - 9 + (y+4)^2 - 16 = 0.$$

Simplifying gives

$$(x-3)^2 + (y+4)^2 = 25.$$

This equation is of the form

$(x - a)^2 + (y - b)^2 = r^2$, with $a = 3$, $b = -4$ and $r^2 = 25$. So it represents a circle, with centre $(3, -4)$ and radius 5.

- (b) The given equation is

$$4x^2 + 4y^2 - 16x + 4y = 3.$$

Dividing through by 4 gives

$$x^2 + y^2 - 4x + y = \frac{3}{4}.$$

Completing the squares gives

$$(x - 2)^2 - 4 + (y + \frac{1}{2})^2 - \frac{1}{4} = \frac{3}{4}.$$

Simplifying gives

$$(x - 2)^2 + (y + \frac{1}{2})^2 = 5.$$

This equation is of the form

$(x - a)^2 + (y - b)^2 = r^2$, with $a = 2$, $b = -\frac{1}{2}$ and $r^2 = 5$. So it represents a circle, with centre $(2, -\frac{1}{2})$ and radius $\sqrt{5}$.

Solution to Activity 8

- (a) The equation $(x - 2)^2 - (y + 3)^2 = 4$ does not represent a circle, because when the brackets are expanded, the coefficient of x^2 is 1, but the coefficient of y^2 is -1 , which is different.
- (b) The equation $(x + 1)^2 + (y - 2)^2 + 9 = 0$ does not represent a circle. When it is written in standard form, the term on the right-hand side is -9 , which is not the square of the radius of a circle.
- (c) The equation $(x + 1)^2 + (y - 3)^2 - 5 = 0$ can be written as $(x + 1)^2 + (y - 3)^2 = 5$, which is the equation of a circle with centre $(-1, 3)$ and radius $\sqrt{5}$.
- (d) The given equation is
- $$2x^2 + 2y^2 - 20x + 16y + 90 = 0.$$
- First simplify it by dividing by 2 to give
- $$x^2 + y^2 - 10x + 8y + 45 = 0.$$
- Then completing the squares gives
- $$(x - 5)^2 - 25 + (y + 4)^2 - 16 + 45 = 0,$$
- which simplifies to
- $$(x - 5)^2 + (y + 4)^2 = -4.$$
- This equation does not represent a circle, since the number on the right-hand side is negative.
- (e) The given equation is
- $$x^2 + y^2 - 10x - 2y + 20 = 0.$$

Completing the squares gives

$$(x - 5)^2 - 25 + (y - 1)^2 - 1 + 20 = 0,$$

which simplifies to

$$(x - 5)^2 + (y - 1)^2 = 6.$$

This equation represents a circle with centre $(5, 1)$ and radius $\sqrt{6}$.

- (f) In the equation $2x^2 + 3y^2 - 5x + 4y - 8 = 0$, the coefficients of x^2 and y^2 are different, so this equation does not represent a circle.

Solution to Activity 9

- (a) Substituting $y = 3$ into the equation of the circle gives

$$(x - 1)^2 + (3 - 5)^2 = 9$$

$$(x - 1)^2 + 4 = 9$$

$$(x - 1)^2 = 5.$$

Taking the square root of both sides gives

$$x - 1 = \pm\sqrt{5}$$

$$x = 1 \pm \sqrt{5}$$

$$x = 1 + \sqrt{5} \quad \text{or} \quad x = 1 - \sqrt{5}.$$

So the points $(1 + \sqrt{5}, 3)$ and $(1 - \sqrt{5}, 3)$ lie on the circle. (One lies on the right-hand part, and one on the left-hand part.)

- (b) Substituting $x = 4$ into the equation of the circle gives

$$(4 - 1)^2 + (y - 5)^2 = 9$$

$$9 + (y - 5)^2 = 9$$

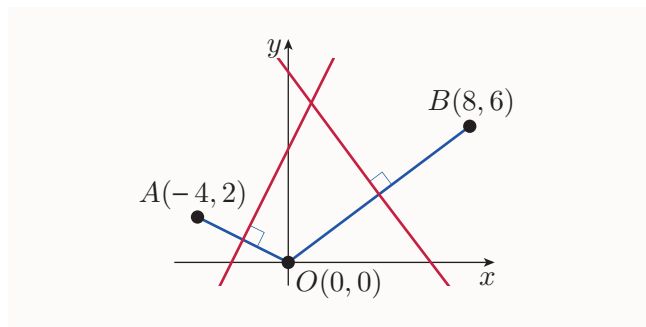
$$(y - 5)^2 = 0$$

$$y = 5.$$

So $(4, 5)$ is the only point on the circle with x -coordinate 4. (It is the point at the extreme right of the circle.)

Solution to Activity 10

Let the three points be $O(0, 0)$, $A(-4, 2)$ and $B(8, 6)$.



First, find the perpendicular bisector of OA . The midpoint of OA is

$$\left(\frac{0-4}{2}, \frac{0+2}{2}\right) = (-2, 1).$$

The gradient of OA is

$$\frac{2-0}{-4-0} = \frac{2}{-4} = -\frac{1}{2}.$$

So the gradient of the perpendicular bisector of OA is 2.

Hence its equation is

$$y - 1 = 2(x - (-2))$$

$$y - 1 = 2x + 4$$

$$y = 2x + 5.$$

Second, find the perpendicular bisector of OB . The midpoint of OB is

$$\left(\frac{0+8}{2}, \frac{0+6}{2}\right) = (4, 3).$$

The gradient of OB is

$$\frac{6-0}{8-0} = \frac{6}{8} = \frac{3}{4}.$$

So the gradient of the perpendicular bisector of OB is $-\frac{4}{3}$.

Hence its equation is

$$y - 3 = -\frac{4}{3}(x - 4)$$

$$y - 3 = -\frac{4}{3}x + \frac{16}{3}$$

$$y = -\frac{4}{3}x + \frac{25}{3}.$$

Now find the point of intersection of the perpendicular bisectors, by solving $y = 2x + 5$ and $y = -\frac{4}{3}x + \frac{25}{3}$ simultaneously.

Equating these expressions for y gives

$$2x + 5 = -\frac{4}{3}x + \frac{25}{3}$$

$$6x + 15 = -4x + 25$$

$$10x = 10$$

$$x = 1.$$

Using $y = 2x + 5$ gives $y = 2 \times 1 + 5 = 7$.

So the centre of the circle is $(1, 7)$.

The radius is the distance between the centre, $(1, 7)$, and one of the given points, say, $(0, 0)$. So it is

$$\sqrt{(1-0)^2 + (7-0)^2} = \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}.$$

Hence the equation of the circle is

$$(x-1)^2 + (y-7)^2 = 50.$$

Solution to Activity 11

We need to find the centre of the circle passing through $(0, 0)$, $(6, 1)$ and $(9, \frac{5}{2})$.

To do this, first find the perpendicular bisector of the line segment that joins $(0, 0)$ and $(6, 1)$. The midpoint of this line segment is

$$\left(\frac{0+6}{2}, \frac{0+1}{2}\right) = \left(3, \frac{1}{2}\right).$$

Its gradient is

$$\frac{1-0}{6-0} = \frac{1}{6}.$$

So the gradient of the perpendicular bisector is -6 , and hence its equation is

$$y - \frac{1}{2} = -6(x - 3)$$

$$y = -6x + 18 + \frac{1}{2}$$

$$y = -6x + \frac{37}{2}.$$

Next, find the perpendicular bisector of the line segment that joins $(6, 1)$ and $(9, \frac{5}{2})$. The midpoint of this line segment is

$$\left(\frac{6+9}{2}, \frac{1+5/2}{2}\right) = \left(\frac{15}{2}, \frac{7}{4}\right).$$

Its gradient is

$$\frac{5/2-1}{9-6} = \frac{3/2}{3} = \frac{1}{2}.$$

So the gradient of the perpendicular bisector is -2 , and hence its equation is

$$y - \frac{7}{4} = -2 \left(x - \frac{15}{2} \right)$$

$$y - \frac{7}{4} = -2x + 15$$

$$y = -2x + \frac{67}{4}.$$

Now find the point of intersection of the perpendicular bisectors by solving the equations

$$y = -6x + \frac{37}{2} \quad \text{and} \quad y = -2x + \frac{67}{4},$$

simultaneously.

Equating the expressions for y gives

$$-6x + \frac{37}{2} = -2x + \frac{67}{4}$$

$$-4x = -\frac{7}{4}$$

$$x = \frac{7}{16}.$$

Substituting into the equation $y = -2x + \frac{67}{4}$ gives

$$y = -2 \times \frac{7}{16} + \frac{67}{4} = -\frac{7}{8} + \frac{67}{4} = \frac{127}{8}.$$

So the centre of the circle is

$$\left(\frac{7}{16}, \frac{127}{8} \right) = (0.4375, 15.875).$$

The radius of the plate is the distance between the centre $(0.4375, 15.875)$ and one of the given points, say $(0, 0)$, which is

$$\sqrt{(0.4375 - 0)^2 + (15.875 - 0)^2} = 15.881 \dots$$

So the radius of the plate is 15.9 cm, and its diameter is 31.8 cm, both to 1 d.p.

Solution to Activity 12

- (a) Substituting the equation of the line, $y = 2x$, into the equation of the circle,
- $$(x + 7)^2 + (y - 2)^2 = 80, \text{ gives}$$

$$(x + 7)^2 + (2x - 2)^2 = 80.$$

Expanding the brackets and simplifying gives

$$5x^2 + 6x - 27 = 0.$$

This factorises as $(x + 3)(5x - 9) = 0$.

So $x + 3 = 0$, that is, $x = -3$, or $5x - 9 = 0$, that is, $x = \frac{9}{5}$.

Substituting each of these values in turn into the equation of the line, we obtain, respectively,

$$y = 2 \times (-3) = -6 \text{ and } y = 2 \times \frac{9}{5} = \frac{18}{5}.$$

So the line intersects the circle at the two points $(-3, -6)$ and $(\frac{9}{5}, \frac{18}{5})$.

- (b) Substituting the equation of the line, $y = 2x - 6$, into the equation of the circle, $(x + 7)^2 + (y - 2)^2 = 80$, gives

$$(x + 7)^2 + (2x - 8)^2 = 80.$$

This equation simplifies to $5x^2 - 18x + 33 = 0$, which is a quadratic equation of the form $ax^2 + bx + c = 0$, with $a = 5$, $b = -18$ and $c = 33$. The discriminant of the quadratic expression is

$$b^2 - 4ac = (-18)^2 - 4 \times 5 \times 33 = -336.$$

Since this is negative, the quadratic equation has no real solutions.

So there are no points which lie on both the line and the circle: the line does not intersect the circle.

Solution to Activity 13

The two equations are $y = x + 11$ and $y = -x^2 - 2x + 15$. Their graphs intersect when

$$-x^2 - 2x + 15 = x + 11$$

$$x^2 + 3x - 4 = 0$$

$$(x + 4)(x - 1) = 0$$

$$x = -4 \quad \text{or} \quad x = 1.$$

We now substitute these solutions into the equation $y = x + 11$ to find the corresponding y -coordinates.

Substituting $x = -4$ gives

$$y = -4 + 11 = 7.$$

Substituting $x = 1$ gives

$$y = 1 + 11 = 12.$$

So the points of intersection are $(-4, 7)$ and $(1, 12)$.

Solution to Activity 14

The equilibrium price is the value of p at the point (q, p) where the supply and demand curves intersect. At this point, we have

$$500 - 30q = \frac{q^2}{2} + \frac{q}{2}$$

$$1000 - 60q = q^2 + q$$

$$q^2 + 61q - 1000 = 0.$$

Unit 5 Coordinate geometry and vectors

Using the quadratic formula with $a = 1$, $b = 61$ and $c = -1000$ gives

$$\begin{aligned} q &= \frac{-61 \pm \sqrt{61^2 - 4 \times 1 \times (-1000)}}{2} \\ &= \frac{-61 \pm \sqrt{7721}}{2} \\ &= 13.434 \dots \quad \text{or} \quad -74.434 \dots \end{aligned}$$

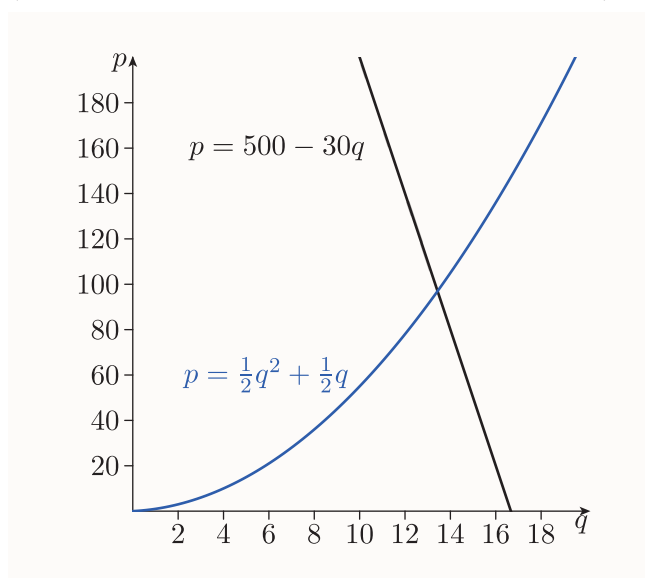
Hence the amount of wheat expected to be produced in the next year is 13 million tonnes (to the nearest million tonnes). (We ignore the negative solution of the equation since the amount of wheat is positive.)

Substituting into the equation $p = 500 - 30q$ gives

$$p = 500 - 30 \times 13.434 \dots = 96.961 \dots$$

So the trader expects that 13 million tonnes of wheat will be produced (to the nearest million tonnes), and that it will sell at £97 per tonne (to the nearest pound).

(The graphs of the functions are shown below.)



Solution to Activity 15

Expanding the equations and simplifying gives

$$x^2 + 2x + 1 + y^2 - 4y + 4 = 9$$

$$x^2 - 8x + 16 + y^2 + 6y + 9 = 36;$$

that is,

$$x^2 + y^2 + 2x - 4y - 4 = 0,$$

$$x^2 + y^2 - 8x + 6y - 11 = 0.$$

Subtracting equation (8) from equation (7) to eliminate the terms in x^2 and y^2 gives

$$10x - 10y + 7 = 0$$

$$10y = 10x + 7,$$

$$\text{so } y = x + \frac{7}{10}.$$

Using this equation to substitute for y in equation (7) gives

$$x^2 + \left(x + \frac{7}{10}\right)^2 + 2x - 4\left(x + \frac{7}{10}\right) - 4 = 0$$

$$x^2 + x^2 + \frac{7}{5}x + \frac{49}{100} + 2x - 4x - \frac{14}{5} - 4 = 0$$

$$2x^2 - \frac{3}{5}x - \frac{631}{100} = 0.$$

Using the quadratic formula, and writing the fractions as exact decimals, gives

$$x = \frac{0.6 \pm \sqrt{0.6^2 + 4 \times 2 \times 6.31}}{4}$$

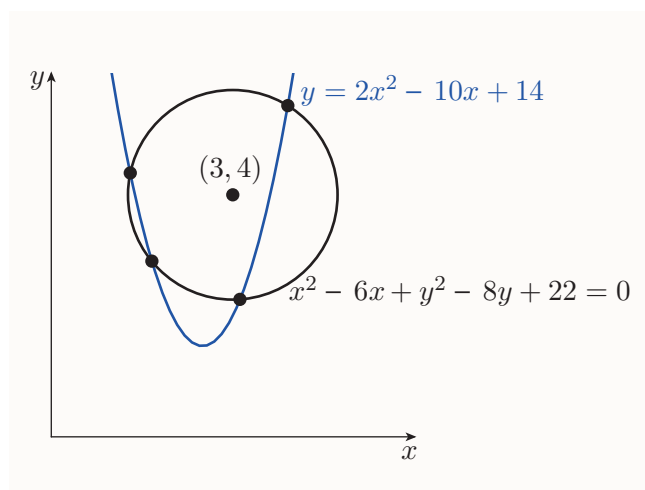
$$x = 1.932 \dots \quad \text{or} \quad -1.632 \dots$$

Substituting these values of x into the equation $y = x + \frac{7}{10}$ gives $y = 2.632 \dots$ and $y = -0.932 \dots$, respectively.

So the points of intersection are $(1.93, 2.63)$ and $(-1.63, -0.93)$ (to two decimal places).

Solution to Activity 17

(a) The curves are plotted below.



- (b) The points of intersection are $(1.30, 4.36)$, $(1.66, 2.90)$, $(3.12, 2.27)$ and $(3.91, 5.47)$ (to 2 d.p.).
- (c) The maximum number of points of intersection of a circle and a parabola is four. This fact can be justified by the geometric argument that each

'half' of the parabola (each part on one side of the vertex) can meet the circle at most twice.

Alternatively, if the equation of the parabola is used to substitute for y in the equation of the circle, then an equation for x containing terms in x^4 , x^3 , x^2 , x and a constant term is obtained, and such an equation can have at most four solutions (by a result that you will meet in Unit 12).

(Details of how to use the CAS to find the points of intersection of two curves are given in the *Computer algebra guide*, in the 'Computer methods for the CAS Activities in Books A–D' section.)

Solution to Activity 18

The points are $A(2, 1, 0)$, $B(4, 4, 0)$, $C(4, 1, 4)$ and $D(2, 4, 4)$.

Solution to Activity 19

- (a) The distance between the points $(1, 3, 5)$ and $(4, 1, 7)$ is

$$\begin{aligned} & \sqrt{(4-1)^2 + (1-3)^2 + (7-5)^2} \\ &= \sqrt{3^2 + (-2)^2 + 2^2} \\ &= \sqrt{9+4+4} = \sqrt{17}. \end{aligned}$$

- (b) The distance between the points $(-1, 2, 5)$ and $(4, -1, 3)$ is

$$\begin{aligned} & \sqrt{(4-(-1))^2 + (-1-2)^2 + (3-5)^2} \\ &= \sqrt{5^2 + (-3)^2 + (-2)^2} \\ &= \sqrt{25+9+4} = \sqrt{38}. \end{aligned}$$

Solution to Activity 20

- (a) The sphere with centre $(3, 5, 2)$ and radius 2 has equation $(x-3)^2 + (y-5)^2 + (z-2)^2 = 4$.

- (b) The sphere with centre $(2, -1, -3)$ and radius $\sqrt{3}$ has equation

$$(x-2)^2 + (y-(-1))^2 + (z-(-3))^2 = 3;$$

that is,

$$(x-2)^2 + (y+1)^2 + (z+3)^2 = 3.$$

Solution to Activity 21

- (a) The sphere $(x-2)^2 + (y-3)^2 + (z-5)^2 = 16$ has centre $(2, 3, 5)$ and radius 4.
- (b) The sphere $(x+2)^2 + (y-4)^2 + (z+1)^2 = 7$ has centre $(-2, 4, -1)$ and radius $\sqrt{7}$.

Solution to Activity 22

- (a) This equation can be written as

$$(x-4)^2 + (y-2)^2 + (z+5)^2 = 3,$$

so it is the equation of a sphere with centre $(4, 2, -5)$ and radius $\sqrt{3}$.

- (b) This equation does not represent a sphere. It has all the terms involving x , y and z on the same side of the equation, but the coefficients of x^2 , y^2 and z^2 are not the same.

- (c) Dividing this equation through by 2 gives

$$x^2 + y^2 + z^2 + 2x - 6z + 6 = 0,$$

and completing the squares gives

$$(x+1)^2 + y^2 + (z-3)^2 = 4,$$

which is the equation of a sphere with centre $(-1, 0, 3)$ and radius 2.

- (d) This equation does not represent a sphere.

Dividing the equation through by 2 gives

$$x^2 + y^2 + z^2 + 4x - 2y + \frac{13}{2} = 0,$$

and completing the squares gives

$$(x+2)^2 + (y-1)^2 + z^2 = -\frac{3}{2}.$$

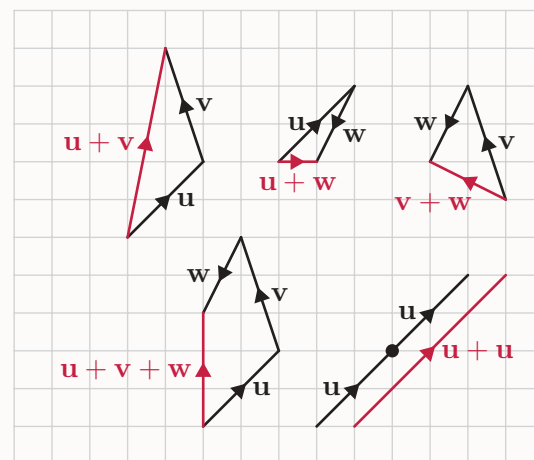
Since the constant term on the right-hand side is negative, this is not the equation of a sphere.

Solution to Activity 23

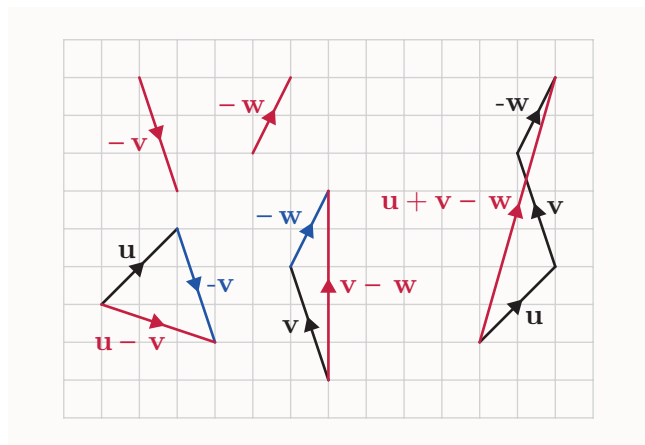
- (a) The vector $\mathbf{f}$ is equal to the vector $\mathbf{a}$.

- (b) The vector $\mathbf{d}$ is equal to the vector $\overrightarrow{PQ}$.

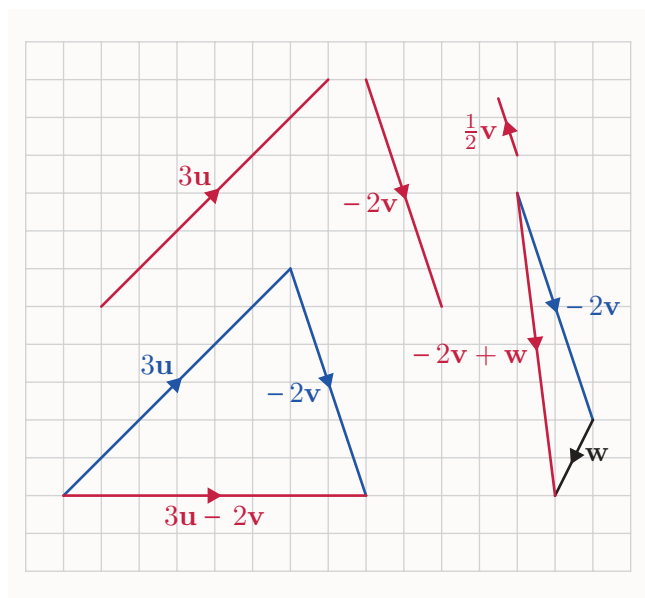
Solution to Activity 24



Solution to Activity 25



Solution to Activity 26



Solution to Activity 27

- (a) The velocity of a wind of 70 knots blowing from the north-east is represented by $2\mathbf{v}$.
- (b) The velocity of a wind of 35 knots blowing from the south-west is represented by $-\mathbf{v}$.

Solution to Activity 28

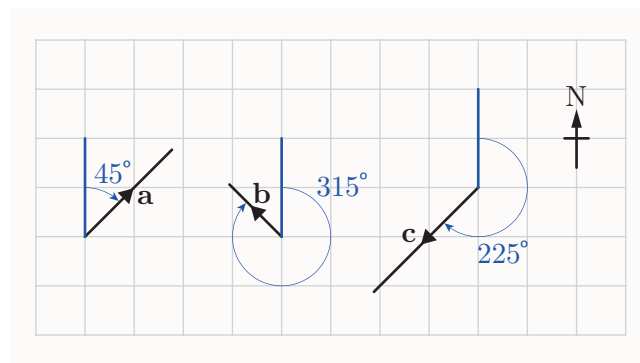
- (a) $4(\mathbf{a} - \mathbf{c}) + 3(\mathbf{c} - \mathbf{b}) + 2(2\mathbf{a} - \mathbf{b} - 3\mathbf{c})$
 $= 4\mathbf{a} - 4\mathbf{c} + 3\mathbf{c} - 3\mathbf{b} + 4\mathbf{a} - 2\mathbf{b} - 6\mathbf{c}$
 $= 8\mathbf{a} - 5\mathbf{b} - 7\mathbf{c}$

- (b) (i) $4\mathbf{x} = 7\mathbf{a} - 2\mathbf{b}$
 $\mathbf{x} = \frac{7}{4}\mathbf{a} - \frac{1}{2}\mathbf{b}$

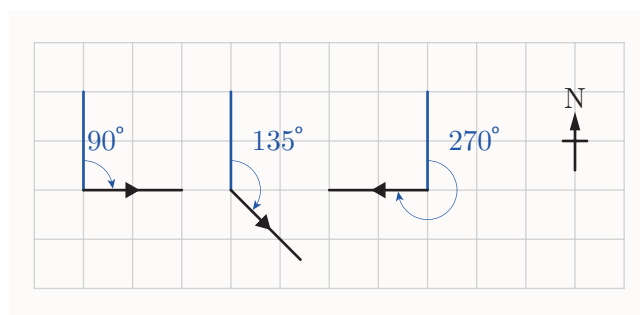
- (ii) $5\mathbf{x} = 2(\mathbf{a} - \mathbf{b}) - 3(\mathbf{b} - \mathbf{a})$
 $= 2(\mathbf{a} - \mathbf{b}) + 3(\mathbf{a} - \mathbf{b})$
 $= 5(\mathbf{a} - \mathbf{b})$
 $\mathbf{x} = \mathbf{a} - \mathbf{b}$

Solution to Activity 29

- (a) The bearing of $\mathbf{a}$ is 45° , the bearing of $\mathbf{b}$ is 315° , and the bearing of $\mathbf{c}$ is 225° , as shown below.

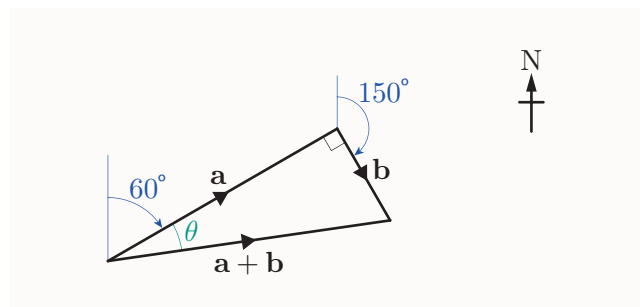


- (b)



Solution to Activity 30

Represent the first part of the motion by the vector $\mathbf{a}$, and the second part by the vector $\mathbf{b}$. Then the resultant displacement is $\mathbf{a} + \mathbf{b}$.



Since $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, and $|\mathbf{a}| = 5.3$ km and $|\mathbf{b}| = 2.1$ km,

$$\begin{aligned} |\mathbf{a} + \mathbf{b}| &= \sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2} \\ &= \sqrt{5.3^2 + 2.1^2} = \sqrt{32.5} \\ &= 5.70 \dots \text{ km.} \end{aligned}$$

The angle marked θ is given by

$$\tan \theta = \frac{|\mathbf{b}|}{|\mathbf{a}|} = \frac{2.1}{5.3},$$

so

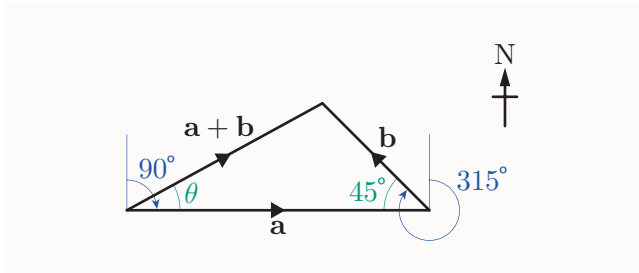
$$\theta = \tan^{-1} \left(\frac{2.1}{5.3} \right) = 22^\circ \text{ (to the nearest degree).}$$

So the bearing of $\mathbf{a} + \mathbf{b}$ is $60^\circ + 22^\circ = 82^\circ$ (to the nearest degree).

The resultant displacement of the yacht has magnitude 5.7 km (to 1 d.p.) and bearing 82° (to the nearest degree).

Solution to Activity 31

Represent the first part of the motion by the vector $\mathbf{a}$, and the second part by the vector $\mathbf{b}$. Then the resultant displacement is $\mathbf{a} + \mathbf{b}$.



The angle at the tip of $\mathbf{a}$ in the triangle above is $315^\circ - 270^\circ = 45^\circ$, as shown.

We know that $|\mathbf{a}| = 40$ cm and $|\mathbf{b}| = 20$ cm.

The magnitude of the resultant $\mathbf{a} + \mathbf{b}$ can be found by using the cosine rule:

$$|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2 \times |\mathbf{a}| \times |\mathbf{b}| \times \cos 45^\circ,$$

which gives

$$\begin{aligned} |\mathbf{a} + \mathbf{b}| &= \sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2 \times |\mathbf{a}| \times |\mathbf{b}| \times \cos 45^\circ} \\ &= \sqrt{40^2 + 20^2 - 2 \times 40 \times 20 \times \cos 45^\circ} \\ &= 29.472 \dots \text{ cm.} \end{aligned}$$

The angle marked θ can be found by using the sine rule:

$$\begin{aligned} \frac{|\mathbf{b}|}{\sin \theta} &= \frac{|\mathbf{a} + \mathbf{b}|}{\sin 45^\circ} \\ \frac{20}{\sin \theta} &= \frac{29.472 \dots}{\sin 45^\circ} \end{aligned}$$

$$\sin \theta = \frac{20 \sin 45^\circ}{29.472 \dots}.$$

Now,

$$\sin^{-1} \left(\frac{20 \sin 45^\circ}{29.472 \dots} \right) = 28.67 \dots^\circ,$$

so

$$\theta = 28.67 \dots^\circ$$

or

$$\theta = 180^\circ - 28.67 \dots^\circ = 151.32 \dots^\circ.$$

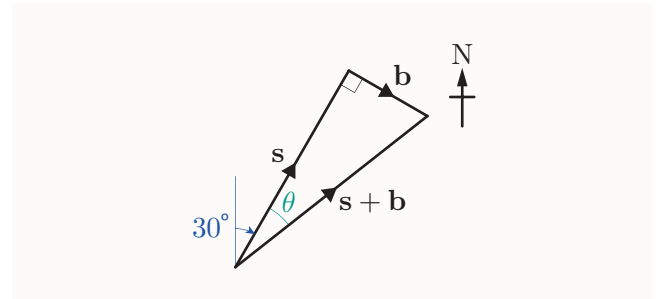
Since $|\mathbf{b}| < |\mathbf{a} + \mathbf{b}|$, we expect $\theta < 45^\circ$, so the required solution is $\theta = 29^\circ$ (to the nearest degree).

The bearing of $|\mathbf{a} + \mathbf{b}|$ is $90^\circ - 29^\circ = 61^\circ$ (to the nearest degree).

So the resultant displacement of the grab has magnitude 29 cm (to the nearest cm) and bearing 61° (to the nearest degree).

Solution to Activity 32

Let $\mathbf{s}$ be the velocity of the ship, and let $\mathbf{b}$ be the velocity of the boy relative to the ship. Then the resultant velocity of the boy is $\mathbf{s} + \mathbf{b}$, as shown below.



We know that $|\mathbf{s}| = 10.0 \text{ m s}^{-1}$ and $|\mathbf{b}| = 4.0 \text{ m s}^{-1}$. Since the triangle is right-angled,

$$\begin{aligned} |\mathbf{s} + \mathbf{b}| &= \sqrt{|\mathbf{s}|^2 + |\mathbf{b}|^2} \\ &= \sqrt{10^2 + 4^2} \\ &= \sqrt{116} \\ &= 10.77 \dots \text{ m s}^{-1}. \end{aligned}$$

The angle θ is given by

$$\tan \theta = \frac{|\mathbf{b}|}{|\mathbf{s}|} = \frac{4}{10} = \frac{2}{5}.$$

So

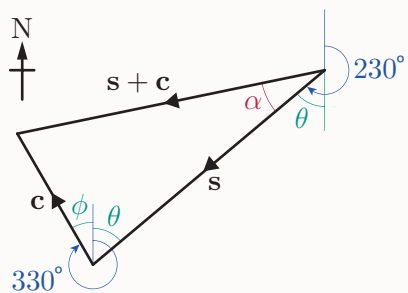
$$\theta = \tan^{-1} \frac{2}{5} = 21.8 \dots^\circ.$$

The bearing of $\mathbf{s} + \mathbf{b}$ is $30^\circ + 21.8 \dots^\circ = 51.8 \dots^\circ$.

So the resultant velocity of the boy is 10.8 m s^{-1} (to 1 d.p.) on a bearing of 52° (to the nearest degree).

Solution to Activity 33

Let $\mathbf{s}$ be the velocity of the ship in still water, and let $\mathbf{c}$ be the velocity of the current. The resultant velocity of the ship is $\mathbf{s} + \mathbf{c}$, as shown below.



We know that $|\mathbf{s}| = 5.7 \text{ m s}^{-1}$ and $|\mathbf{c}| = 2.5 \text{ m s}^{-1}$.

The angle marked θ at the tail of $\mathbf{s}$ is given by $\theta = 230^\circ - 180^\circ = 50^\circ$. Since alternate angles are equal, the angle θ marked at the tip of $\mathbf{s}$ has the same size.

The angle ϕ marked at the tail of $\mathbf{c}$ is given by $\phi = 360^\circ - 330^\circ = 30^\circ$.

So the bottom angle of the triangle is $\theta + \phi = 50^\circ + 30^\circ = 80^\circ$.

Applying the cosine rule gives

$$|\mathbf{s} + \mathbf{c}|^2 = |\mathbf{s}|^2 + |\mathbf{c}|^2 - 2|\mathbf{s}||\mathbf{c}|\cos(\theta + \phi),$$

so

$$|\mathbf{s} + \mathbf{c}| = \sqrt{5.7^2 + 2.5^2 - 2 \times 5.7 \times 2.5 \times \cos 80^\circ} = 5.813 \dots \text{ m s}^{-1}.$$

The angle α can be found by using the sine rule:

$$\frac{|\mathbf{c}|}{\sin \alpha} = \frac{|\mathbf{s} + \mathbf{c}|}{\sin(\theta + \phi)}$$

$$\sin \alpha = \frac{|\mathbf{c}| \sin(\theta + \phi)}{|\mathbf{s} + \mathbf{c}|} = \frac{2.5 \sin 80^\circ}{5.813 \dots}$$

Now,

$$\sin^{-1} \left(\frac{2.5 \sin 80^\circ}{5.813 \dots} \right) = 25.058 \dots^\circ.$$

So $\alpha = 25.058 \dots^\circ$ or

$$\alpha = 180^\circ - 25.058 \dots^\circ = 154.941 \dots^\circ.$$

But $|\mathbf{c}| < |\mathbf{s} + \mathbf{c}|$, so we expect $\alpha < \theta + \phi$; that is, $\alpha < 80^\circ$. So $\alpha = 25.058 \dots^\circ$, and hence the bearing of $\mathbf{s} + \mathbf{c}$ is $230^\circ + 25.058 \dots^\circ = 255.058 \dots^\circ$.

The resultant velocity of the ship is 5.8 m s^{-1} (to 1 d.p.) on a bearing of 255° (to the nearest degree).

Solution to Activity 34

The component forms are $\mathbf{p} = 0\mathbf{i} + 3\mathbf{j} = 3\mathbf{j}$, $\mathbf{q} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j}$.

Solution to Activity 35

The column forms of these vectors are

$$\mathbf{p} = 0\mathbf{i} + 3\mathbf{j} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad \mathbf{q} = 3\mathbf{i} + 4\mathbf{j} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ and}$$

$$\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Solution to Activity 36

$$(a) \quad (4\mathbf{i} - 2\mathbf{j}) + (-3\mathbf{i} + \mathbf{j}) = 4\mathbf{i} - 3\mathbf{i} - 2\mathbf{j} + \mathbf{j} = \mathbf{i} - \mathbf{j}$$

$$(b) \quad \begin{pmatrix} 5 \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \end{pmatrix} = \begin{pmatrix} 5-4 \\ 3-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(c) \quad \begin{pmatrix} -7 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -7+2+5 \\ -4+7+1 \\ 1-3+4 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$$

Solution to Activity 37

$$(a) \quad (2\mathbf{i} + \mathbf{j}) - (3\mathbf{i} + 2\mathbf{j})$$

$$= 2\mathbf{i} + \mathbf{j} - 3\mathbf{i} - 2\mathbf{j} = -\mathbf{i} - \mathbf{j}$$

$$(b) \quad (3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) - (-2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$$

$$= 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + 2\mathbf{i} - 4\mathbf{j} - 2\mathbf{k} = 5\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$$

$$(c) \quad \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3-2 \\ 4-(-1) \\ -1-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$$

Solution to Activity 38

$$(a) \quad 4\mathbf{a} = 4 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$$

$$(b) \quad -2\mathbf{a} = -2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

$$(c) \quad \frac{1}{2}\mathbf{a} = \frac{1}{2} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

$$(d) \quad 3\mathbf{b} = 3(\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) = 3\mathbf{i} + 9\mathbf{j} - 18\mathbf{k}$$

$$(e) \quad -4\mathbf{b} = -4(\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) = -4\mathbf{i} - 12\mathbf{j} + 24\mathbf{k}$$

$$(f) \quad \frac{1}{3}\mathbf{b} = \frac{1}{3}(\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) = \frac{1}{3}\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

Solution to Activity 39

- (a) $-2\mathbf{a} + 3\mathbf{b} + 4\mathbf{c}$
 $= -2(2\mathbf{i} + 3\mathbf{j}) + 3(\mathbf{i} - 4\mathbf{j}) + 4(-5\mathbf{i} + 7\mathbf{j})$
 $= -4\mathbf{i} - 6\mathbf{j} + 3\mathbf{i} - 12\mathbf{j} - 20\mathbf{i} + 28\mathbf{j}$
 $= -21\mathbf{i} + 10\mathbf{j}$
- (b) $2\begin{pmatrix} 6 \\ -3 \end{pmatrix} - 7\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 5\begin{pmatrix} -1 \\ 4 \end{pmatrix}$
 $= \begin{pmatrix} 2 \times 6 - 7 \times 1 + 5 \times (-1) \\ 2 \times (-3) - 7 \times 2 + 5 \times 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- (c) $a_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
 $= \begin{pmatrix} a_1 \times 1 + a_2 \times 0 \\ a_1 \times 0 + a_2 \times 1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$
- (d) $2\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$
 $= \begin{pmatrix} 2 \times 1 + 3 \times 0 - 2 \times 1 \\ 2 \times 3 + 3 \times 3 - 2 \times 4 \\ 2 \times 4 + 3 \times 1 - 2 \times 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 7 \end{pmatrix}$

Solution to Activity 40

In component form, $\overrightarrow{OA} = 5\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{OB} = -2\mathbf{i} + 4\mathbf{j}$, so
 $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
 $= -2\mathbf{i} + 4\mathbf{j} - (5\mathbf{i} + 3\mathbf{j})$
 $= -7\mathbf{i} + \mathbf{j}.$

Solution to Activity 41

- (a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}.$
- (b) Since both OAB and OBC are equilateral triangles, $\angle ABO = \angle BOC = 60^\circ$. So OC is parallel to AB , and $\overrightarrow{OC} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}.$
- (c) $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (\mathbf{b} - \mathbf{a}) - \mathbf{b} = -\mathbf{a}.$
 Alternatively, notice that BC is parallel to OA , so $\overrightarrow{BC} = -\overrightarrow{OA} = -\mathbf{a}.$
- (d) OD is parallel to OA and of equal length, so $\overrightarrow{AD} = -2\overrightarrow{OA} = -2\mathbf{a}.$
- (e) $\overrightarrow{AE} = \overrightarrow{AO} + \overrightarrow{OE} = -\mathbf{a} - \mathbf{b}.$

Solution to Activity 42

- (a) The point C is the midpoint of OB , so its position vector is $\frac{1}{2}\mathbf{b}$. Hence $\overrightarrow{AC} = \frac{1}{2}\mathbf{b} - \mathbf{a}.$
 The point D is the midpoint of OA , so its position vector is $\frac{1}{2}\mathbf{a}$. Hence $\overrightarrow{BD} = \frac{1}{2}\mathbf{a} - \mathbf{b}.$
- (b) The point E is the midpoint of AB , so its position vector is $\frac{1}{2}(\mathbf{a} + \mathbf{b})$. Hence the position vector of the point $\frac{2}{3}$ of the way along OE from O is
 $\frac{2}{3} \times \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{3}(\mathbf{a} + \mathbf{b}).$
 The position vector of the point $\frac{2}{3}$ of the way along AC from A is
 $\overrightarrow{OA} + \frac{2}{3}\overrightarrow{AC} = \mathbf{a} + \frac{2}{3} \times (\frac{1}{2}\mathbf{b} - \mathbf{a})$
 $= \mathbf{a} + \frac{1}{3}\mathbf{b} - \frac{2}{3}\mathbf{a}$
 $= \frac{1}{3}(\mathbf{a} + \mathbf{b}).$
 The position vector of the point $\frac{2}{3}$ of the way along BD from B is
 $\overrightarrow{OB} + \frac{2}{3}\overrightarrow{BD} = \mathbf{b} + \frac{2}{3} \times (\frac{1}{2}\mathbf{a} - \mathbf{b})$
 $= \mathbf{b} + \frac{1}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}$
 $= \frac{1}{3}(\mathbf{a} + \mathbf{b}).$
- (c) The three points in part (b) all have the same position vector, so they are all the same point. Hence the lines OE , AC and BD are concurrent.

Solution to Activity 43

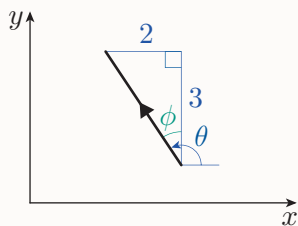
- (a) $|-3\mathbf{i} + \mathbf{j}| = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$
- (b) $|2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}| = \sqrt{2^2 + 4^2 + (-3)^2} = \sqrt{29}$
- (c) $\left| \begin{pmatrix} -2 \\ 0 \end{pmatrix} \right| = \sqrt{(-2)^2 + 0^2} = 2$
- (d) $\left| \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \right| = \sqrt{3^2 + (-1)^2 + 3^2} = \sqrt{19}$

Solution to Activity 44

- (a) The magnitude of the vector $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j}$ is
 $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$

Unit 5 Coordinate geometry and vectors

The vector **a** is shown below.



The angle ϕ is given by

$$\phi = \tan^{-1} \frac{2}{3} = 34^\circ \text{ (to the nearest degree).}$$

So the angle θ that **a** makes with the positive x -direction is $34^\circ + 90^\circ = 124^\circ$ (to the nearest degree).

(b) The magnitude of the vector $\mathbf{b} = \begin{pmatrix} 0 \\ -3.5 \end{pmatrix}$ is

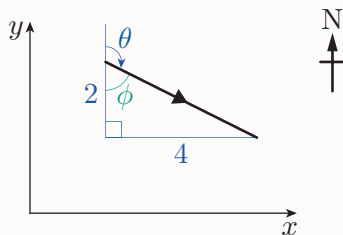
$$|\mathbf{b}| = \sqrt{0^2 + (-3.5)^2} = 3.5.$$

The vector **b** has **i**-component 0, so it is parallel to the y -axis. Its **j**-component is negative, so it points in the negative y -direction. Hence the angle that it makes with the positive x -direction is 270° .

Solution to Activity 45

(a) The magnitude of the vector $\mathbf{u} = 4\mathbf{i} - 2\mathbf{j}$ is

$$\begin{aligned} |\mathbf{u}| &= \sqrt{4^2 + (-2)^2} \\ &= \sqrt{20} = 4.5 \text{ m s}^{-1} \text{ (to 1 d.p.).} \end{aligned}$$



The angle ϕ is given by

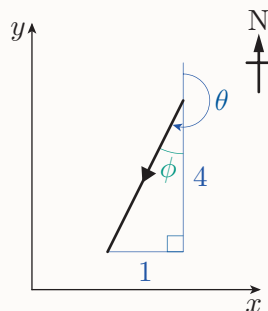
$$\phi = \tan^{-1} \left(\frac{4}{2} \right) = \tan^{-1} 2 = 63.43 \dots^\circ.$$

So the bearing of **u** is

$\theta = 180^\circ - 63.43 \dots^\circ = 117^\circ$ (to the nearest degree).

(b) The magnitude of the vector $\mathbf{v} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$ is

$$\begin{aligned} |\mathbf{v}| &= \sqrt{(-1)^2 + (-4)^2} \\ &= \sqrt{17} = 4.1 \text{ m s}^{-1} \text{ (to 1 d.p.).} \end{aligned}$$



The angle ϕ is given by

$$\phi = \tan^{-1} \frac{1}{4} = 14.03 \dots^\circ.$$

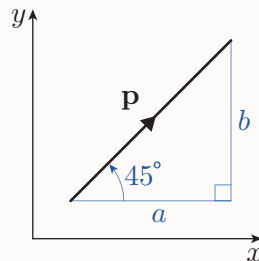
So the bearing of **v** is

$\theta = 180^\circ + 14.03 \dots^\circ = 194^\circ$ (to the nearest degree).

Solution to Activity 46

In each case, we draw a diagram and calculate the lengths of the shorter sides of the right-angled triangle whose hypotenuse is the vector, and whose shorter sides are parallel to the axes.

(a)



The side lengths a and b are given by

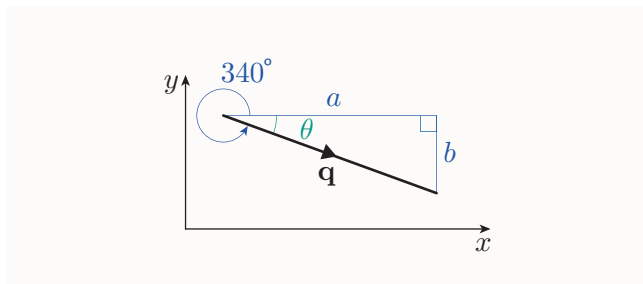
$$a = |\mathbf{p}| \cos 45^\circ = 1 \times \cos 45^\circ = 0.7 \text{ (to 1 d.p.)}$$

$$b = |\mathbf{p}| \sin 45^\circ = 1 \times \sin 45^\circ = 0.7 \text{ (to 1 d.p.)}$$

The diagram shows that both components of **p** are positive, so

$$\mathbf{p} = 0.7\mathbf{i} + 0.7\mathbf{j} \text{ (to 1 d.p.).}$$

(b)



The acute angle marked θ in the diagram is given by $\theta = 360^\circ - 340^\circ = 20^\circ$. Hence the side lengths a and b are given by

$$a = |\mathbf{q}| \cos \theta = 2.5 \cos 20^\circ = 2.3 \text{ (to 1 d.p.)}$$

$$b = |\mathbf{q}| \sin \theta = 2.5 \sin 20^\circ = 0.9 \text{ (to 1 d.p.)}$$

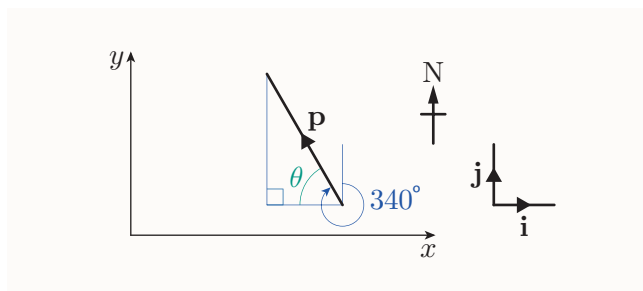
The diagram shows that the $\mathbf{i}$ -component is positive and the $\mathbf{j}$ -component is negative, so

$$\mathbf{q} = 2.3\mathbf{i} - 0.9\mathbf{j} \text{ (to 1 d.p.)}$$

Solution to Activity 47

In each case, we draw a diagram of the right-angled triangle whose hypotenuse is the vector, and whose shorter sides are parallel to the axes. The required components are equal to the lengths of the shorter sides, with their signs deduced from the diagram.

(a)



The angle θ is given by $\theta = 340^\circ - 270^\circ = 70^\circ$. Hence the $\mathbf{i}$ - and $\mathbf{j}$ -components of $\mathbf{p}$ are

$$-|\mathbf{p}| \cos \theta = -3.5 \cos 70^\circ = -1.2 \text{ (to 1 d.p.)}$$

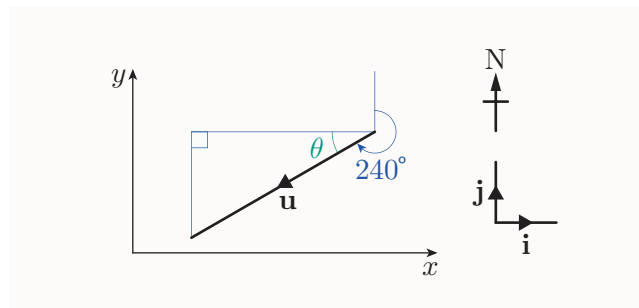
and

$$|\mathbf{p}| \sin \theta = 3.5 \sin 70^\circ = 3.3 \text{ (to 1 d.p.)},$$

respectively. Hence

$$\mathbf{p} = -1.2\mathbf{i} + 3.3\mathbf{j} \text{ (in m, to 1 d.p.)}$$

(b)



The angle θ is given by $\theta = 270^\circ - 240^\circ = 30^\circ$. Hence the $\mathbf{i}$ - and $\mathbf{j}$ -components of $\mathbf{u}$ are

$$-|\mathbf{u}| \cos \theta = -5.2 \cos 30^\circ = -4.5 \text{ (to 1 d.p.)}$$

and

$$-|\mathbf{u}| \sin \theta = -5.2 \sin 30^\circ = -2.6,$$

respectively. Hence

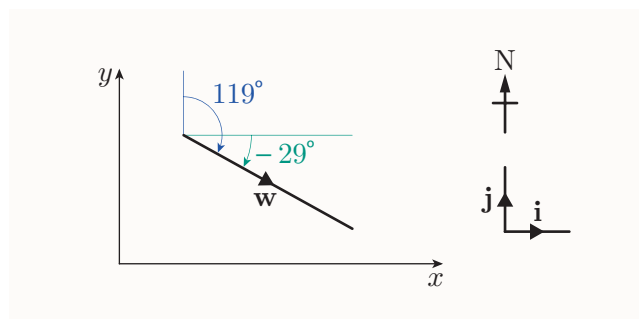
$$\mathbf{u} = -4.5\mathbf{i} - 2.6\mathbf{j} \text{ (in m s}^{-1}\text{, to 1 d.p.)}$$

Solution to Activity 48

(a) By the formula for the components in the box before the activity,

$$\begin{aligned} \mathbf{a} &= 78 \cos 216^\circ \mathbf{i} + 78 \sin 216^\circ \mathbf{j} \\ &= -63.103 \dots \mathbf{i} - 45.847 \dots \mathbf{j} \\ &= -63\mathbf{i} - 46\mathbf{j} \text{ (to 2 s.f.)} \end{aligned}$$

(b) The vector $\mathbf{w}$ has bearing 119° , so it makes an angle of $-(119^\circ - 90^\circ) = -29^\circ$ with the positive x -direction, as shown below.



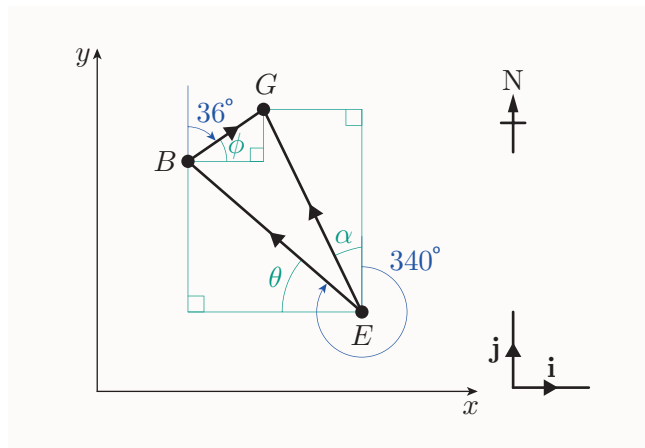
Hence, by the formula for components,

$$\begin{aligned} \mathbf{w} &= 4.4 \cos(-29^\circ) \mathbf{i} + 4.4 \sin(-29^\circ) \mathbf{j} \\ &= 3.848 \dots \mathbf{i} - 2.133 \dots \mathbf{j} \\ &= 3.8\mathbf{i} - 2.1\mathbf{j} \text{ (in m s}^{-1}\text{, to 2 s.f.)} \end{aligned}$$

Solution to Activity 49

Denote Exeter by E , Belfast by B , and Glasgow by G .

We know that $EB = 460$ km and $BG = 173$ km.



Choose a coordinate system as shown.

The angle θ at E is given by $\theta = 340^\circ - 270^\circ = 70^\circ$, so

$$\begin{aligned}\vec{EB} &= -|\vec{EB}| \cos \theta \mathbf{i} + |\vec{EB}| \sin \theta \mathbf{j} \\ &= -460 \cos 70^\circ \mathbf{i} + 460 \sin 70^\circ \mathbf{j} \\ &= -157.329 \dots \mathbf{i} + 432.258 \dots \mathbf{j}.\end{aligned}$$

The angle ϕ at B is given by $\phi = 90^\circ - 36^\circ = 54^\circ$, so

$$\begin{aligned}\vec{BG} &= |\vec{BG}| \cos \phi \mathbf{i} + |\vec{BG}| \sin \phi \mathbf{j} \\ &= 173 \cos 54^\circ \mathbf{i} + 173 \sin 54^\circ \mathbf{j} \\ &= 101.686 \dots \mathbf{i} + 139.959 \dots \mathbf{j}.\end{aligned}$$

Hence

$$\begin{aligned}\vec{EG} &= \vec{EB} + \vec{BG} \\ &= -157.329 \dots \mathbf{i} + 432.258 \dots \mathbf{j} \\ &\quad + 101.686 \dots \mathbf{i} + 139.959 \dots \mathbf{j} \\ &= -55.642 \dots \mathbf{i} + 572.218 \dots \mathbf{j}.\end{aligned}$$

The magnitude of $\vec{EG}$ is

$$|\vec{EG}| = \sqrt{(-55.642 \dots)^2 + (572.218 \dots)^2} = 575 \text{ km}$$

(to the nearest km).

The direction of $\vec{EG}$ can be found from the angle α in the diagram. We have

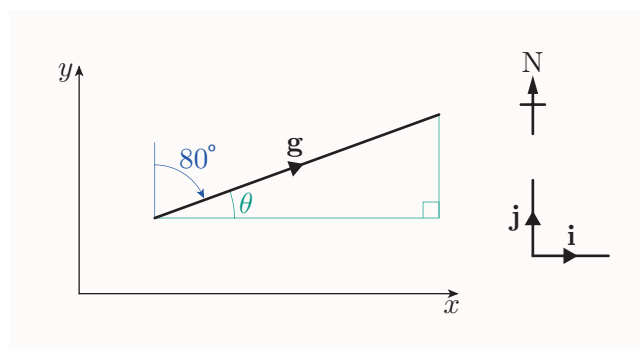
$$\alpha = \tan^{-1} \left(\frac{55.642 \dots}{572.218 \dots} \right) = 5.55 \dots^\circ = 6^\circ$$

(to the nearest degree). Hence the bearing of $\vec{EG}$ is $360^\circ - 6^\circ = 354^\circ$ (to the nearest degree).

So the magnitude of the displacement of Glasgow from Exeter is 575 km (to the nearest km) and the bearing is 354° (to the nearest degree).

Solution to Activity 50

- (a) Let the horizontal ground velocity of the aircraft be the vector $\mathbf{g}$. A sketch of $\mathbf{g}$ is shown below.



The angle θ in the diagram is given by $\theta = 90^\circ - 80^\circ = 10^\circ$.

The magnitude of $\mathbf{g}$ is 600 mph, so

$$\begin{aligned}\mathbf{g} &= |\mathbf{g}| \cos \theta \mathbf{i} + |\mathbf{g}| \sin \theta \mathbf{j} \\ &= 600 \cos 10^\circ \mathbf{i} + 600 \sin 10^\circ \mathbf{j} \\ &= 590.88 \dots \mathbf{i} + 104.18 \dots \mathbf{j} \\ &= 591 \mathbf{i} + 104 \mathbf{j} \text{ (to the nearest mph)}.\end{aligned}$$

- (b) Let the resultant velocity of the aircraft be the vector $\mathbf{t}$. The vertical component of $\mathbf{t}$ is $30\mathbf{k}$, so

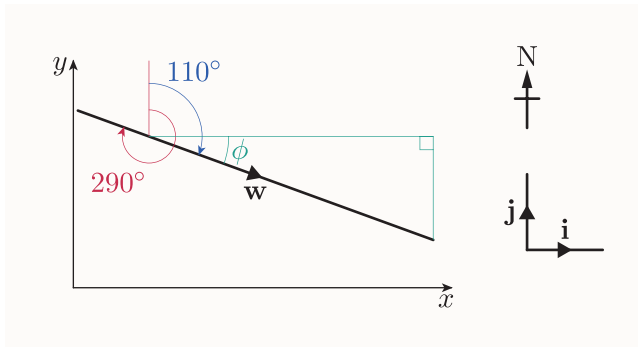
$$\begin{aligned}\mathbf{t} &= \mathbf{g} + 30\mathbf{k} \\ &= 590.88 \dots \mathbf{i} + 104.18 \dots \mathbf{j} + 30\mathbf{k} \\ &= 591 \mathbf{i} + 104 \mathbf{j} + 30 \mathbf{k} \text{ (to the nearest mph)}.\end{aligned}$$

Hence the magnitude of the resultant velocity is

$$\begin{aligned}|\mathbf{t}| &= \sqrt{(590.88 \dots)^2 + (104.18 \dots)^2 + 30^2} \\ &= 600.74 \dots = 601 \text{ mph (to the nearest mph)}.\end{aligned}$$

- (c) Let the air velocity of the aircraft be $\mathbf{a}$, and the wind velocity be $\mathbf{w}$. Then $\mathbf{t} = \mathbf{a} + \mathbf{w}$, so $\mathbf{a} = \mathbf{t} - \mathbf{w}$.

We express $\mathbf{w}$ in component form, by first drawing a diagram. The wind is blowing *from* the direction with a bearing of 290° , so it is blowing *towards* the direction with a bearing of $290^\circ - 180^\circ = 110^\circ$, as shown in the following diagram.



The angle ϕ is given by $\phi = 110^\circ - 90^\circ = 20^\circ$.

The magnitude of $\mathbf{w}$ is 100 mph, so

$$\begin{aligned}\mathbf{w} &= |\mathbf{w}| \cos \phi \mathbf{i} - |\mathbf{w}| \sin \phi \mathbf{j} \\ &= 100 \cos 20^\circ \mathbf{i} - 100 \sin 20^\circ \mathbf{j} \\ &= 93.96 \dots \mathbf{i} - 34.20 \dots \mathbf{j}.\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{a} &= \mathbf{t} - \mathbf{w} \\ &= 590.88 \dots \mathbf{i} + 104.18 \dots \mathbf{j} + 30\mathbf{k} \\ &\quad - (93.96 \dots \mathbf{i} - 34.20 \dots \mathbf{j}) \\ &= 496.91 \dots \mathbf{i} + 138.39 \dots \mathbf{j} + 30\mathbf{k} \\ &= 497\mathbf{i} + 138\mathbf{j} + 30\mathbf{k} \text{ (to the nearest mph)}.\end{aligned}$$

The magnitude of the air velocity is

$$\begin{aligned}|\mathbf{a}| &= \sqrt{(496.91 \dots)^2 + (138.39 \dots)^2 + 30^2} \\ &= 516.69 \dots = 517 \text{ mph (to the nearest mph)}.\end{aligned}$$

Solution to Activity 51

These scalar products are

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= |\mathbf{u}| |\mathbf{v}| \cos \theta = 4 \times 3 \times \cos 60^\circ = 6 \\ \mathbf{u} \cdot \mathbf{w} &= |\mathbf{u}| |\mathbf{w}| \cos \theta = 4 \times 2 \times \cos 90^\circ = 0 \\ \mathbf{u} \cdot \mathbf{u} &= |\mathbf{u}| |\mathbf{u}| \cos \theta = 4 \times 4 \times \cos 0 = 16.\end{aligned}$$

Solution to Activity 52

$$\begin{aligned}(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) &= \mathbf{a} \cdot (\mathbf{a} - \mathbf{b}) + \mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} \\ &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{b} \\ &= \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b}\end{aligned}$$

Solution to Activity 53

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (3\mathbf{i} - 4\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \\ &= 3 \times 2 + (-4) \times 3 + 1 \times (-2) \\ &= 6 - 12 - 2 = -8\end{aligned}$$

$$\begin{aligned}\mathbf{u} \cdot \mathbf{w} &= (3\mathbf{i} - 4\mathbf{j} + \mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \\ &= 3 \times (-1) + (-4) \times 1 + 1 \times 3 \\ &= -3 - 4 + 3 = -4\end{aligned}$$

$$\begin{aligned}\mathbf{v} \cdot \mathbf{w} &= (2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \\ &= 2 \times (-1) + 3 \times 1 + (-2) \times 3 \\ &= -2 + 3 - 6 = -5\end{aligned}$$

Solution to Activity 54

We have

$$\mathbf{a} \cdot \mathbf{b} = 2 \times (-1) + (-3) \times 2 + 1 \times 4 = -4,$$

$$|\mathbf{a}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14},$$

$$|\mathbf{b}| = \sqrt{(-1)^2 + 2^2 + 4^2} = \sqrt{21}.$$

So, if θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{-4}{\sqrt{14} \times \sqrt{21}} = -\frac{4}{7\sqrt{6}}$$

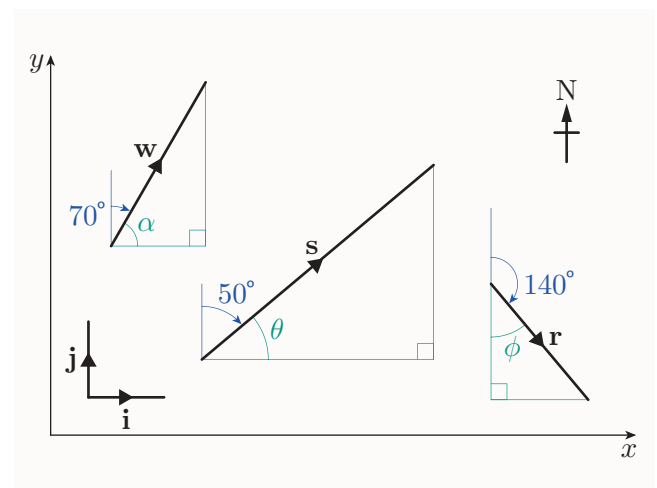
and hence

$$\theta = \cos^{-1} \left(-\frac{4}{7\sqrt{6}} \right) = 103.49 \dots^\circ.$$

So the angle between the vectors is 103° (to the nearest degree).

Solution to Activity 55

- (a) The initial angle between the paths of the boats is the difference between their courses, $140^\circ - 50^\circ = 90^\circ$.
- (b) Let $\mathbf{s}$ be the initial velocity of the yacht, let $\mathbf{r}$ be the initial velocity of the rowing boat, and let $\mathbf{w}$ be the velocity imposed by the wind. These vectors are shown below. A coordinate system with $\mathbf{i}$ pointing east and $\mathbf{j}$ pointing north has been chosen, as shown.



We express the three vectors $\mathbf{s}$, $\mathbf{r}$ and $\mathbf{w}$ in component form.

The angle θ is given by $\theta = 90^\circ - 50^\circ = 40^\circ$, so

$$\begin{aligned}\mathbf{s} &= |\mathbf{s}| \cos \theta \mathbf{i} + |\mathbf{s}| \sin \theta \mathbf{j} \\ &= 4.8 \cos 40^\circ \mathbf{i} + 4.8 \sin 40^\circ \mathbf{j} \\ &= 3.677 \dots \mathbf{i} + 3.085 \dots \mathbf{j}.\end{aligned}$$

The angle ϕ is given by $\phi = 180^\circ - 140^\circ = 40^\circ$, so

$$\begin{aligned}\mathbf{r} &= |\mathbf{r}| \sin \phi \mathbf{i} - |\mathbf{r}| \cos \phi \mathbf{j} \\ &= 1.1 \sin 40^\circ \mathbf{i} - 1.1 \cos 40^\circ \mathbf{j} \\ &= 0.707 \dots \mathbf{i} - 0.842 \dots \mathbf{j}.\end{aligned}$$

The angle α is given by $\alpha = 90^\circ - 70^\circ = 20^\circ$, so

$$\begin{aligned}\mathbf{w} &= |\mathbf{w}| \cos \alpha \mathbf{i} + |\mathbf{w}| \sin \alpha \mathbf{j} \\ &= 3.5 \cos 20^\circ \mathbf{i} + 3.5 \sin 20^\circ \mathbf{j} \\ &= 3.288 \dots \mathbf{i} + 1.197 \dots \mathbf{j}.\end{aligned}$$

The new velocity of the yacht is

$$\begin{aligned}\mathbf{s} + \mathbf{w} &= (3.677 \dots \mathbf{i} + 3.085 \dots \mathbf{j}) + (3.288 \dots \mathbf{i} + 1.197 \dots \mathbf{j}) \\ &= 6.965 \dots \mathbf{i} + 4.282 \dots \mathbf{j}.\end{aligned}$$

Hence its speed is

$$\sqrt{(6.965 \dots)^2 + (4.282 \dots)^2} = 8.177 \dots = 8.2 \text{ m s}^{-1} \text{ (to 1 d.p.)}.$$

The new velocity of the rowing boat is

$$\begin{aligned}\mathbf{r} + \mathbf{w} &= (0.707 \dots \mathbf{i} - 0.842 \dots \mathbf{j}) + (3.288 \dots \mathbf{i} + 1.197 \dots \mathbf{j}) \\ &= 3.995 \dots \mathbf{i} + 0.354 \dots \mathbf{j}.\end{aligned}$$

Hence its speed is

$$\sqrt{(3.995 \dots)^2 + (0.354 \dots)^2} = 4.011 \dots = 4.0 \text{ m s}^{-1} \text{ (to 1 d.p.)}.$$

Let β be the angle between the new velocities,

$\mathbf{s} + \mathbf{w}$ and $\mathbf{r} + \mathbf{w}$. Then

$$\cos \beta = \frac{(\mathbf{s} + \mathbf{w}) \cdot (\mathbf{r} + \mathbf{w})}{|\mathbf{s} + \mathbf{w}| |\mathbf{r} + \mathbf{w}|}.$$

We have

$$\begin{aligned}(\mathbf{s} + \mathbf{w}) \cdot (\mathbf{r} + \mathbf{w}) &= (6.965 \dots \mathbf{i} + 4.282 \dots \mathbf{j}) \cdot (3.995 \dots \mathbf{i} + 0.354 \dots \mathbf{j}) \\ &= (6.965 \dots) \times (3.995 \dots) + (4.282 \dots) \times (0.354 \dots) \\ &= 29.353 \dots\end{aligned}$$

So

$$\begin{aligned}\beta &= \cos^{-1} \left(\frac{(\mathbf{s} + \mathbf{w}) \cdot (\mathbf{r} + \mathbf{w})}{|\mathbf{s} + \mathbf{w}| |\mathbf{r} + \mathbf{w}|} \right) \\ &= \cos^{-1} \left(\frac{29.353 \dots}{(8.177 \dots) \times (4.011 \dots)} \right) \\ &= 26.5 \dots^\circ = 27^\circ \text{ (to the nearest degree)}.\end{aligned}$$

So the new angle between the paths of the boats is 27° (to the nearest degree).

So, in summary, the new speeds of the yacht and the rowing boat are 8.2 m s^{-1} and 4.0 m s^{-1} , respectively (both to one decimal place), and the new angle between their paths is 27° (to the nearest degree).

Acknowledgements

Grateful acknowledgement is made to the following sources:

Figure 23: Taken from:

<http://furthermathematicst.blogspot.co.uk/2011/07/121-3d-vectors.html>

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<http://en.wikipedia.org/wiki/File:Hybrid-bicycle-1.jpg>.

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http://en.wikipedia.org/wiki/File:02Compass_baseplate_2010-03-09.jpg.

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